FROM HYPERCOMPLEX TO HOLOMORPHIC SYMPLECTIC STRUCTURES

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Abstract. The notions of holomorphic symplectic structures and hypercomplex structures on Courant algebroids are introduced and then proved to be equivalent. These generalize hypercomplex triples and holomorphic symplectic 2-forms on manifolds respectively. Basic properties of such structures are established.

Contents

1. Introduction

This paper is an extension of [16]. Here, we make the case that, when seen in the framework of Courant algebroids, hypercomplex structures and holomorphic symplectic structures are one and the same concept.

A *hypercomplex manifold* is a smooth manifold M endowed with three complex structures i, j, k (regarded as endomorphisms of the tangent bundle of M) that satisfy the quaternionic relations $i^2 = j^2 = k^2 = ijk = -1$. A characteristic feature of hypercomplex manifolds discovered by Obata early on is the existence of a unique torsion-free connection ∇ that satisfies $\nabla i = \nabla j = \nabla k = 0$ [13]. Hypercomplex manifolds have been the subject of much attention in the past. Noteworthy are the constructions of left-invariant hypercomplex structures on compact Lie groups and homogeneous spaces due to Spindel, Sevrin, Troos & Van Proeyen (in 1988) and also to Joyce (in 1992). Moreover, important examples of hypercomplex manifolds arose in mathematical physics in the form of hyper-Kähler manifolds.[1]

A *holomorphic symplectic manifold* is a complex manifold $(M; j)$ endowed with a closed non-degenerate holomorphic 2-form ω. Hyper-Kähler manifolds, which carry three symplectic 2-forms

[1]Hyper-Kähler manifolds are hypercomplex manifolds $(M; i, j, k)$ endowed with a Riemannian metric g with respect to which i, j, and k are covariantly constant and mutually orthogonal.

each of which is holomorphic with respect to one of the three complex structures, constitute again a special subclass.

The generalized complex geometry introduced in the last decade by Hitchin [8] and Gualtieri [5] provides the motivation for attempting to unify hypercomplex and holomorphic symplectic structures. A generalized complex structure on a manifold M is an endomorphism J of the vector bundle $TM \oplus T^*M$, orthogonal with respect to a natural symmetric pairing, and satisfying $J^2 = -1$ and $\mathcal{N}(J, J) = 0$, where $\mathcal{N}$ denotes the Nijenhuis concomitant of a pair of endomorphisms of the Courant algebroid $TM \oplus T^*M$. A generalized complex structure on a manifold M can thus be seen as a complex structure on the corresponding (standard) Courant algebroid $TM \oplus T^*M$. Complex structures have been defined on arbitrary Courant algebroids in a similar fashion [11, 14].

Three new concepts are introduced in the present paper. They generalize hypercomplex manifolds, the Obata connection, and holomorphic symplectic 2-forms to the realm of Courant algebroids:

(1) A *hypercomplex structure on a Courant algebroid* E is defined as a triple of complex structures I, J, K on E satisfying the quaternionic relations $I^2 = J^2 = K^2 = IJK = -1$. Hypercomplex manifolds, holomorphic symplectic 2-forms, and hyper-Poisson manifolds provide particular examples. The notion of hyper-Poisson structure, also introduced in this paper, can be seen as a degenerate analogue of hyper-Kähler structures.
(2) The analogue of the Obata connection for a Courant algebroid E endowed with a hypercomplex triple (I, J, K) is called a *hypercomplex connection.* Though a hypercomplex connection is not itself a connection in the usual sense, its restrictions to all Dirac subbundles of E stable under I, J, K are torsion-free (Lie algebroid) connections.
(3) A *holomorphic symplectic structure on a Courant algebroid* E *relative to a complex structure* J *on* E is a section Ω of $\wedge^2 L_J$ such that $\Omega^\sharp \overline{\Omega}^\sharp = -\operatorname{id}_{L_J}$ (‘nondegeneracy’) and $d_{L_J^*}\Omega = 0$ (‘closedness’). Here L_J and L_J^* denote the eigenbundles of J. Given a complex manifold $(M; j)$, let $J = \left(\begin{smallmatrix} j & 0 \\ 0 & -j^* \end{smallmatrix}\right)$ be the corresponding complex structure on the standard Courant algebroid $TM \oplus T^*M$. The holomorphic symplectic structures on $(TM \oplus T^*M; J)$ are instances of extended Poisson structures in the sense of [3].

We prove the following three theorems:

(1) A Courant algebroid endowed with a hypercomplex structure admits a unique hypercomplex connection (see Theorems 3.13 and 3.14).
(2) There exists a one-to-one correspondence between the hypercomplex structures and the holomorphic symplectic structures on a Courant algebroid (see Theorem 4.6).
(3) Given a holomorphic symplectic structure Ω on a Courant algebroid E relative to a complex structure J on E with eigenbundles L_J and L_J^*, the restriction of the hypercomplex connection on E to any Lie subalgebroid of L_J^* maximal isotropic with respect to Ω is a flat torsion-free (Lie algebroid) connection (see Theorem 5.5).

Finally, given a complex Lagrangian foliation of a complex manifold $(M; j)$ endowed with a holomorphic symplectic 2-form ω, we apply the third result above to the special case in which $E = TM \oplus T^*M$, $J = \left(\begin{smallmatrix} j & 0 \\ 0 & -j^* \end{smallmatrix}\right)$, and $\Omega = \omega + \overline{\omega}^{-1}$, and thereby recover a connection on the Lagrangian foliation, as discovered by Behrend & Fantechi [2].

2. Complex structures on Courant algebroids

A *Courant algebroid* (see [11, 14]) consists of a vector bundle $\pi : E \to M$, a nondegenerate symmetric pairing $\langle , \rangle$ on the fibers of π, a bundle map $\rho : E \to TM$ called the anchor, and an $\mathbb{R}$-bilinear operation $\circ$ on $\Gamma(E)$ called the Dorfman bracket, which, for all $f \in C^\infty(M)$ and

$x, y, z \in \Gamma(E)$, satisfy the relations

$$\begin{gathered} x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z), \\ \rho(x \circ y) = [\rho(x), \rho(y)], \\ x \circ fy = (\rho(x)f)y + f(x \circ y), \\ x \circ y + y \circ x = 2\mathcal{D}\langle x, y\rangle, \\ \mathcal{D}f \circ x = 0, \\ \rho(x)\langle y, z\rangle = \langle x \circ y, z\rangle + \langle y, x \circ z\rangle, \end{gathered}$$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is the $\mathbb{R}$-linear map defined by $\langle \mathcal{D}f, x\rangle = \frac{1}{2}\rho(x)f$.

There is a canonical isomorphism $E \xrightarrow{\Psi} E^*$ given by $\Psi(e_1) : e_2 \mapsto \langle e_1, e_2\rangle$ for any $e_1, e_2 \in E$ and an induced isomorphism $\wedge^k E \xrightarrow{\Psi} \wedge^k E^*$. Sometimes, we will implicitly identify $\wedge^k E$ and $\wedge^k E^*$ in this paper.

T. Courant described the following *standard example* in [4]. Given a smooth manifold M, the vector bundle $TM \oplus T^*M \to M$ carries a natural Courant algebroid structure: the anchor map is the projection onto the tangent component, whereas the pairing and the Dorfman bracket are respectively given by

$$\begin{aligned} & \langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)) \\ \text{and} \qquad & (X + \xi) \circ (Y + \eta) = [X, Y] + (L_X\eta - \iota_Y d\xi), \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$.

Let $(E, \rho, \langle,\rangle, \circ)$ be a Courant algebroid. Given two endomorphisms F and G of the vector bundle E, their *Nijenhuis concomitant* $\mathcal{N}(F, G) : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$ is defined as

$$\begin{aligned} \mathcal{N}(F, G)(U, V) =& FU \circ GV - F(U \circ GV) - G(FU \circ V) + FG(U \circ V) \\ & + GU \circ FV - G(U \circ FV) - F(GU \circ V) + GF(U \circ V), \end{aligned}$$

where $U, V \in \Gamma(E)$ (see [16]). Obviously, $\mathcal{N}(F, G) = \mathcal{N}(G, F)$. In addition, we define an $\mathbb{R}$-trilinear map $\mathcal{N}_{F,G} : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \to C^\infty(M)$ by

$$\mathcal{N}_{F,G}(U, V, W) = \langle \mathcal{N}(F, G)(U, V), W\rangle.$$

Remark 2.1. It is easy to verify that $\mathcal{N}(F, G)(U, fV) = f\mathcal{N}(F, G)(U, V)$ for all $f \in C^\infty(M)$. But usually $\mathcal{N}(F, G)(fU, V) \neq f\mathcal{N}(F, G)(U, V)$; therefore, $\mathcal{N}(F, G)$ is not necessarily a tensor.

Lemma 2.2. *Let $(E, \rho, \langle,\rangle, \circ)$ be a Courant algebroid. If F, G are two skew-symmetric endomorphisms of the vector bundle E such that*

$$FG + GF = \lambda\, \mathrm{id}_E, \quad (\lambda \in \mathbb{R}), \tag{2.1}$$

then $\mathcal{N}_{F,G} \in \Gamma(\wedge^3 E^*)$.

Proof. For all $U, V, W \in \Gamma(E)$, we have

$$\begin{aligned} & \mathcal{N}(F, G)(U, V) + \mathcal{N}(F, G)(V, U) \\ =& 2\mathcal{D}\langle FU, GV\rangle - 2F\mathcal{D}\langle U, GV\rangle - 2G\mathcal{D}\langle FU, V\rangle + 2FG\mathcal{D}\langle U, V\rangle \\ & + 2\mathcal{D}\langle GU, FV\rangle - 2G\mathcal{D}\langle U, FV\rangle - 2F\mathcal{D}\langle GU, V\rangle + 2GF\mathcal{D}\langle U, V\rangle \\ =& - 2\mathcal{D}\langle (FG + GF)U, V\rangle + 2(FG + GF)\mathcal{D}\langle U, V\rangle \\ =& - 2\mathcal{D}\langle \lambda U, V\rangle + 2\lambda\mathcal{D}\langle U, V\rangle \\ =& 0. \end{aligned}$$

Similarly, by a straightforward computation, we prove that

$$\langle \mathcal{N}(F,G)(U,V),W\rangle + \langle \mathcal{N}(F,G)(U,W),V\rangle = 0.$$

This completes the proof. □

Remark 2.3. In the case $F = G$, Lemma 2.2 was proved by Kosmann-Schwarzbach in [9].

Lemma 2.4. *Let $(E, \rho, \langle,\rangle, \circ)$ be a Courant algebroid over a manifold M, F be a skew-symmetric endomorphism of the vector bundle E, and $\pi_F \in \Gamma(\wedge^2 TM)$ be the bivector field defined by*

$$\pi_F(df, dg) = \langle F\mathcal{D}f, \mathcal{D}g\rangle \tag{2.2}$$

for all $f, g \in C^\infty(M)$. Set $\{f, g\} = \pi_F(df, dg) = \langle F\mathcal{D}f, \mathcal{D}g\rangle$. Then

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = -\frac{1}{4}\mathcal{N}_{F,F}(\mathcal{D}f, \mathcal{D}g, \mathcal{D}h),$$

which shows that π_F is a Poisson bivector field on M if $\mathcal{N}(F,F) = 0$.

Proof. For all $f, g, h \in C^\infty(M)$, we have

$$\begin{aligned}
&\{\{f,g\},h\} = \langle F\mathcal{D}\langle F\mathcal{D}f, \mathcal{D}g\rangle, \mathcal{D}h\rangle \\
=&\langle \frac{1}{2}F(F\mathcal{D}f \circ \mathcal{D}g + \mathcal{D}g \circ F\mathcal{D}f), \mathcal{D}h\rangle = \frac{1}{2}\langle F(F\mathcal{D}f \circ \mathcal{D}g), \mathcal{D}h\rangle;
\end{aligned}$$

$$\begin{aligned}
&\{\{g,h\},f\} = \langle F\mathcal{D}\langle F\mathcal{D}g, \mathcal{D}h\rangle, \mathcal{D}f\rangle = \langle \mathcal{D}\langle F\mathcal{D}g, \mathcal{D}h\rangle, -F\mathcal{D}f\rangle \\
=&-\frac{1}{2}\rho(F\mathcal{D}f)\langle F\mathcal{D}g, \mathcal{D}h\rangle = -\frac{1}{2}\langle F\mathcal{D}f \circ F\mathcal{D}g, \mathcal{D}h\rangle - \frac{1}{2}\langle F\mathcal{D}g, F\mathcal{D}f \circ \mathcal{D}h\rangle;
\end{aligned}$$

$$\begin{aligned}
&\{\{h,f\},g\} = \langle F\mathcal{D}\langle F\mathcal{D}h, \mathcal{D}f\rangle, \mathcal{D}g\rangle = \langle \mathcal{D}\langle \mathcal{D}h, F\mathcal{D}f\rangle, F\mathcal{D}g\rangle \\
=&\frac{1}{2}\langle \mathcal{D}h \circ F\mathcal{D}f + F\mathcal{D}f \circ \mathcal{D}h, F\mathcal{D}g\rangle = \frac{1}{2}\langle F\mathcal{D}g, F\mathcal{D}f \circ \mathcal{D}h\rangle.
\end{aligned}$$

Therefore,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = \frac{1}{2}\langle F(F\mathcal{D}f \circ \mathcal{D}g), \mathcal{D}h\rangle - \frac{1}{2}\langle F\mathcal{D}f \circ F\mathcal{D}g, \mathcal{D}h\rangle.$$

On the other hand,

$$\begin{aligned}
&\mathcal{N}_{F,F}(\mathcal{D}f, \mathcal{D}g, \mathcal{D}h) = \langle \mathcal{N}(F,F)(\mathcal{D}f, \mathcal{D}g), \mathcal{D}h\rangle \\
=&\langle 2F\mathcal{D}f \circ F\mathcal{D}g - 2F(F\mathcal{D}f \circ \mathcal{D}g) - 2F(\mathcal{D}f \circ F\mathcal{D}g) - 2F^2(\mathcal{D}f \circ \mathcal{D}g), \mathcal{D}h\rangle \\
=&\langle 2F\mathcal{D}f \circ F\mathcal{D}g - 2F(F\mathcal{D}f \circ \mathcal{D}g), \mathcal{D}h\rangle.
\end{aligned}$$

This completes the proof. □

Definition 2.5. An *almost complex structure on a Courant algebroid* $(E, \rho, \langle,\rangle, \circ)$ is an endomorphism J of the vector bundle E (i.e. a vector bundle map over $\mathrm{id}_M : M \to M$), which is an orthogonal transformation with respect to the pairing $\langle,\rangle$ and satisfies $J^2 = -1$.

Let J be an almost complex structure on a Courant algebroid E, and let L_J (resp. $\overline{L_J}$) be the subbundle of $E_\mathbb{C} = E \otimes \mathbb{C}$ associated with the eigenvalue $i = \sqrt{-1}$ (resp. $-i = -\sqrt{-1}$) of J. Then L_J and $\overline{L}_J$ are maximal isotropic subbundles of $E_\mathbb{C}$, such that $E_\mathbb{C} = L_J \oplus \overline{L}_J$.

The nondegenerate symmetric pairing $\langle,\rangle$ identifies $\overline{L}_J$ to L_J^* in a canonical way. We will, therefore, use the symbols $\overline{L}_J$ and L_J^* interchangeably in this paper.

Definition 2.6 ([8]). A *complex structure on a Courant algebroid* is an almost complex structure J whose Nijenhuis concomitant $\mathcal{N}(J,J)$ vanishes—J is said to be integrable.

It is easy to see that an almost complex structure J is integrable if and only if L_J is involutive, i.e. $\Gamma(L_J)$ is closed under the Dorfman bracket. The following result is an immediate consequence of Lemma 2.4.

Corollary 2.7 ([6, 1]). *If J is a complex structure on a Courant algebroid over a smooth manifold M, then π_J is a Poisson bivector field on M.*

It is known that the pair of eigenbundles (L_J, L_J^*) of a complex structure J on a Courant algebroid forms a Lie bialgebroid in the sense of Mackenzie & Xu [12]. The exterior differentials

$$d_{L_J}:\Gamma(\wedge^k L_J^*)\to\Gamma(\wedge^{k+1}L_J^*) \qquad \text{and} \qquad d_{L_J^*}:\Gamma(\wedge^k L_J)\to\Gamma(\wedge^{k+1}L_J)$$

respectively associated with the Lie algebroids L_J and L_J^* satisfy $d_{L_J}^2=0$ and $d_{L_J^*}^2=0$.

Example 2.8 ([5]). Let j be an almost complex structure on M, and let $E_\phi=(TM\oplus T^*M)_\phi$ be the standard Courant algebroid twisted by a closed 3-form $\phi\in\Omega^3(M)$ (see [15]). Then

$$J=\begin{pmatrix} j & 0\\ 0 & -j^*\end{pmatrix}$$

is a complex structure on $E_\phi=(TM\oplus T^*M)_\phi$ if and only if j is a complex structure on M, and $\phi\in\Omega^{2,1}(M)\oplus\Omega^{1,2}(M)$, where $\Omega^{p,q}(M)$ denotes the space of exterior differential forms of type (p,q) relatively to the complex structure j. These conditions hold when M is a complex surface and ϕ is any closed 3-form on M.

Lemma 2.9 ([10]). *If π_1 is a bivector field on a smooth manifold M and j is an endomorphism of the tangent bundle of M, then*

$$J=\begin{pmatrix} j & \pi_1\\ 0 & -j^*\end{pmatrix}$$

is a complex structure on the standard Courant algebroid $E=TM\oplus T^*M$ if and only if j is a complex structure on M and $\pi_1+\sqrt{-1}\pi_2$, with π_2 defined by $\pi_2^\sharp=-j\pi_1^\sharp=-\pi_1^\sharp j^*$, is a holomorphic Poisson structure with respect to j.

3. Hypercomplex structures on Courant algebroids

3.1. **Hypercomplex structure.**

Definition 3.1. An *almost hypercomplex structure on a Courant algebroid* $(E,\rho,\langle,\rangle,\circ)$ is a triple (I,J,K) of almost complex structures on E satisfying the quaternionic relations

$$I^2=J^2=K^2=IJK=-1.$$

This definition has a few immediate consequences. From $J^2=-1$, it follows that $\frac{1}{2}(1-iJ)$ is the projection of $E\otimes\mathbb{C}=L_J\oplus L_J^*$ onto L_J, whereas $\frac{1}{2}(1+iJ)$ is the projection onto L_J^*. Moreover, the endomorphisms I, J, and K anticommute. Therefore, both I and K swap the subbundles L_J and L_J^*, whereas J preserves them. Finally, the relations

$$\Big(\frac{1+I}{\sqrt{2}}\Big)\Big(\frac{1+iJ}{2}\Big)=\Big(\frac{1+iK}{2}\Big)\Big(\frac{1+I}{\sqrt{2}}\Big)$$
$$\Big(\frac{1-I}{\sqrt{2}}\Big)\Big(\frac{1+iK}{2}\Big)=\Big(\frac{1+iJ}{2}\Big)\Big(\frac{1-I}{\sqrt{2}}\Big)$$

imply the inclusions

$$\Big(\frac{1+I}{\sqrt{2}}\Big)L_J^* \subset L_K^* \qquad \text{and} \qquad \Big(\frac{1-I}{\sqrt{2}}\Big)L_K^* \subset L_J^*.$$

Since $\left(\frac{1-I}{\sqrt{2}}\right)\left(\frac{1+I}{\sqrt{2}}\right) = 1$, we obtain a pair of inverse isomorphisms:

$$L_J^* \xrightarrow{\frac{1+I}{\sqrt{2}}} L_K^* \qquad \text{and} \qquad L_J^* \xleftarrow{\frac{1-I}{\sqrt{2}}} L_K^*.$$

Remark 3.2. Given an almost hypercomplex structure (I, J, K) on a Courant algebroid E, it is easy to see that $\mathcal{N}(I, I)$, $\mathcal{N}(I, J)$, $\mathcal{N}(I, K)$, $\mathcal{N}(J, J)$, $\mathcal{N}(J, K)$, and $\mathcal{N}(K, K)$ are $(2, 1)$-tensors, i.e. vector bundle maps $E \otimes E \to E$ over id_M. We can, therefore, regard $\mathcal{N}_{I,I}$, $\mathcal{N}_{I,J}$, $\mathcal{N}_{I,K}$, $\mathcal{N}_{J,J}$, $\mathcal{N}_{J,K}$, and $\mathcal{N}_{K,K}$ as sections of $E^* \otimes E^* \otimes E^*$.

Lemma 2.2 implies

Lemma 3.3. *If (I, J, K) is an almost hypercomplex structure on a Courant algebroid E, then $\mathcal{N}_{I,I}, \mathcal{N}_{I,J}, \mathcal{N}_{I,K}, \mathcal{N}_{J,J}, \mathcal{N}_{J,K}, \mathcal{N}_{K,K} \in \Gamma(\wedge^3 E^*)$.*

Definition 3.4. A *hypercomplex structure on a Courant algebroid* is an almost hypercomplex structure (I, J, K) such that the Nijenhuis tensors $\mathcal{N}(I, I)$, $\mathcal{N}(I, J)$, $\mathcal{N}(I, K)$, $\mathcal{N}(J, J)$, $\mathcal{N}(J, K)$, and $\mathcal{N}(K, K)$ vanish.

Proposition 3.5. *If (I, J, K) is a hypercomplex structure on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$, then $\lambda_1 I + \lambda_2 J + \lambda_3 K$ is a complex structure on E for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$.*

Corollary 3.6. *If π_I, π_J, π_K are the bivector fields associated with a hypercomplex structure (I, J, K) on a Courant algebroid as in Lemma 2.4, then*

$$[\![\pi_\alpha, \pi_\beta]\!] = 0, \quad \forall \alpha, \beta \in \{I, J, K\}.$$

Proof. For any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ satisfying $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, $\lambda_1\pi_I + \lambda_2\pi_J + \lambda_3\pi_K$ is the Poisson structure associated with the complex structure $\lambda_1 I + \lambda_2 J + \lambda_3 K$ on E, from which the corollary immediately follows. □

Example 3.7. The quaternion algebra $\mathbb{H}$ can be regarded as a Courant algebroid over the one point space with the commutator as bracket and $\{1, i, j, k\}$ as an orthonormal basis. If I, J, and K denote the multiplication by i, j, and k from the left respectively, then (I, J, K) is a hypercomplex structure on the Courant algebroid $(\mathbb{H}, \langle,\rangle, [,])$.

Example 3.8. Let i, j, and k be almost complex structures on a smooth manifold M. Then the triple

$$I = \begin{pmatrix} i & 0 \\ 0 & -i^* \end{pmatrix}, \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}, \qquad K = \begin{pmatrix} k & 0 \\ 0 & -k^* \end{pmatrix}$$

is a hypercomplex structure on $TM \oplus T^*M$ if and only if the triple i, j, k is hypercomplex in the classical sense (see [13] or [18]).

Example 3.9. Let j be a complex structure on a smooth manifold M, and let ω_1 and ω_2 be two nondegenerate 2-forms on M. The triple

$$I = \begin{pmatrix} 0 & \omega_1^{-1} \\ -\omega_1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & \omega_2^{-1} \\ -\omega_2 & 0 \end{pmatrix}$$

is a hypercomplex structure on $TM \oplus T^*M$ if and only if $\omega_1 - \sqrt{-1}\omega_2$ is a holomorphic symplectic 2-form on M. We will discuss this case in more detail in Example 4.16.

Example 3.10. Let (i, j, k) be a hypercomplex structure on a four-dimensional manifold M, and let ϕ be a closed 3-form on M. Then

$$I = \begin{pmatrix} i & 0 \\ 0 & -i^* \end{pmatrix}, \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}, \qquad K = \begin{pmatrix} k & 0 \\ 0 & -k^* \end{pmatrix}$$

is a hypercomplex structure on the twisted Courant algebroid $E_\phi = (TM \oplus T^*M)_\phi$.

3.2. **Hypercomplex connection.** Let us recall a classical result pertaining to hypercomplex manifolds.

Theorem 3.11 (Obata connection [13, 18])**.** (1) *Let M be a manifold endowed with a hypercomplex structure (i, j, k). There exists a unique torsion-free connection ∇ on M such that*

$$\nabla i = \nabla j = \nabla k = 0,$$

which is given by the expression

$$\nabla_X Y = -\frac{1}{2} k([jY, iX] - j[Y, iX] - i[jY, X] + ji[Y, X]), \quad \forall X, Y \in \mathfrak{X}(M).$$

(2) *Conversely, given an almost hypercomplex structure (i, j, k) on M, if there exists a torsion-free connection ∇ on M such that $\nabla i = \nabla j = \nabla k = 0$, then (i, j, k) must be a hypercomplex structure on M.*

We will generalize this result to hypercomplex structures on Courant algebroids. Let (I, J, K) be an almost hypercomplex structure on a Courant algebroid E. For all $f \in C^\infty(M)$ and $U, V \in \Gamma(E)$, set

$$\triangle_f(U, V) = \langle U, V\rangle \mathcal{D}f + \langle IU, V\rangle I\mathcal{D}f + \langle JU, V\rangle J\mathcal{D}f + \langle KU, V\rangle K\mathcal{D}f. \tag{3.1}$$

It is simple to check that

$$\triangle_f(U, IV) = I\triangle_f(U, V),$$
$$\triangle_f(U, JV) = J\triangle_f(U, V),$$
$$\triangle_f(U, KV) = K\triangle_f(U, V),$$

and

$$\triangle_f(U, V) + \triangle_f(V, U) = 2\langle U, V\rangle \mathcal{D}f.$$

Definition 3.12. A *hypercomplex connection* on a Courant algebroid $(E, \rho, \langle , \rangle, \circ)$ endowed with an almost hypercomplex structure (I, J, K) is an $\mathbb{R}$-bilinear map

$$\Gamma(E) \otimes \Gamma(E) \to \Gamma(E), \qquad (U, V) \mapsto \nabla_U V$$

such that

$$\nabla_{fU} V = f\nabla_U V \tag{3.2}$$

and

$$\nabla_U(fV) = (\rho(U)f)V + f(\nabla_U V) - \triangle_f(U, V), \tag{3.3}$$

for all $f \in C^\infty(M)$ and $U, V \in \Gamma(E)$. Its torsion is given by

$$T^\nabla(U, V) = \nabla_U V - \nabla_V U - [\![U, V]\!] \tag{3.4}$$

and its curvature by

$$R^\nabla(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[\![U,V]\!]} W \tag{3.5}$$

for all $U, V, W \in \Gamma(E)$, where $[\![U, V]\!] = \frac{1}{2}(U \circ V - V \circ U)$.

Theorem 3.13 ([16])**.** *Let (I, J, K) be a hypercomplex structure on a Courant algebroid $(E, \rho, \langle , \rangle, \circ)$. There exists a unique hypercomplex connection ∇ that satisfies*

$$\nabla I = \nabla J = \nabla K = 0, \tag{3.6}$$

and

$$T^{\nabla}(U, V) = I\mathcal{D}\langle U, IV\rangle + J\mathcal{D}\langle U, JV\rangle + K\mathcal{D}\langle U, KV\rangle, \quad \forall U, V \in \Gamma(E). \tag{3.7}$$

It is given by

$$\nabla_U V = -\frac{1}{2}K(JV \circ IU - J(V \circ IU) - I(JV \circ U) + JI(V \circ U)), \quad \forall U, V \in \Gamma(E). \tag{3.8}$$

The main result of this section is the following theorem.

Theorem 3.14 ([16])**.** *Let (I, J, K) be an almost hypercomplex structure on a Courant algebroid $(E, \rho, \langle , \rangle, \circ)$. The following assertions are equivalent:*

(1) $\mathcal{N}(I, J) = 0$*;*
(2) $\mathcal{N}(I, I) = \mathcal{N}(J, J) = 0$*;*
(3) *The triple (I, J, K) is a hypercomplex structure: all six Nijenhuis tensors $\mathcal{N}(I, I)$, $\mathcal{N}(I, J)$, $\mathcal{N}(I, K)$, $\mathcal{N}(J, J)$, $\mathcal{N}(J, K)$, and $\mathcal{N}(K, K)$ vanish.*
(4) *There exists a unique hypercomplex connection ∇ that satisfies*

$$\nabla I = \nabla J = \nabla K = 0, \tag{3.9}$$

and, for all $U, V \in \Gamma(E)$,

$$T^{\nabla}(U, V) = I\mathcal{D}\langle U, IV\rangle + J\mathcal{D}\langle U, JV\rangle + K\mathcal{D}\langle U, KV\rangle. \tag{3.10}$$

As an application of Theorem 3.14 for Example 3.9, we have the following corollary.

Corollary 3.15. *Let ω_1 and ω_2 be two nondegenerate forms on a manifold M endowed with an almost complex structure j such that $\omega_2^\sharp = j^*\omega_1^\sharp = \omega_1^\sharp j$. Then any one of the following assertions is a consequence of the other two:*

(1) $d\omega_1 = 0$,
(2) $d\omega_2 = 0$,
(3) *j is integrable.*

From this corollary, we immediately obtain the following standard result as given by Hitchin:

Lemma 3.16 ([7])**.** *Let g be a Riemannian metric on a smooth manifold with skew-adjoint endomorphisms i, j, and k of the tangent bundle satisfying the quaternionic conditions. Then g is hyper-Kähler if and only if the corresponding 2-forms $\omega_1, \omega_2, \omega_3$ are closed. (The 2-forms $\omega_1, \omega_2, \omega_3$ are related to the endomorphisms i, j, k by $\omega_1^\sharp = g^\sharp \circ i$, $\omega_2^\sharp = g^\sharp \circ j$, $\omega_3^\sharp = g^\sharp \circ k$.)*

The isotropic, involutive subbundles of a Courant algebroid are necessarily Lie algebroids. Those of maximal rank are called *Dirac structures.*

Let L be a Lie algebroid with anchor ρ and let V be a vector bundle both over the same smooth manifold M. An L-connection on V is a bilinear map $\nabla : \Gamma(L) \times \Gamma(V) \mapsto \Gamma(V)$ that satisfies

$$\nabla_{fX} v = f\nabla_X v,$$
$$\nabla_X fv = f\nabla_X v + (\rho(X)f)v,$$

for all $X \in \Gamma(L)$, $v \in \Gamma(V)$ and $f \in C^\infty(M)$.

The following lemma is a direct consequence of Theorem 3.14 and Equations (3.1) and (3.2).

Lemma 3.17. *Let ∇ denote the hypercomplex connection defined by Equation* (3.8) *on a Courant algebroid $(E, \rho, \langle , \rangle, \circ)$ endowed with a hypercomplex triple (I, J, K). If L is an isotropic, involutive subbundle of E stable under I, J, and K, then ∇ induces a torsion-free L-connection on L.*

Example 3.18. Let $(M;i,j,k)$ be a hypercomplex manifold; let (I,J,K) be the corresponding hypercomplex structure on the standard Courant algebroid $TM \oplus T^*M$ as in Example 3.8; and let $\mathcal{F} = TS$ be the integrable distribution corresponding to a foliation S. The Dirac subbundle $L = \mathcal{F} \oplus \mathcal{F}^{\perp}$ is stable under I,J,K if and only if $\mathcal{F}$ is stable under i,j,k. In this situation, the hypercomplex connection ∇ defined in Equation (3.8) defines a torsion-free L-connection on L such that $\nabla I = \nabla J = \nabla K = 0$. If we take $\mathcal{F} = TM$, then we get the Obata connection.

Example 3.19. Let (M,j,ω) be a holomorphic symplectic manifold with the complex structure j and the holomorphic symplectic form $\omega = \omega_1 - \sqrt{-1}\omega_2$; let I,J,K be the hypercomplex structure on the standard Courant algebroid $TM \oplus T^*M$ as in Example 3.9; and let $\mathcal{F} = TS$ be the integrable distribution of a foliation S. Then the Dirac subbundle $L = \mathcal{F} \oplus \mathcal{F}^{\perp}$ is stable under I,J,K if and only if S is a complex Lagrangian foliation of $(M;j,\omega)$. In this situation, the hypercomplex connection ∇ defined in Equation (3.8) defines a torsion-free L-connection on L that satisfies $\nabla I = \nabla J = \nabla K = 0$. Explicitly, $\forall X,Y \in \Gamma(\mathcal{F})$, the connection can be written as

$$\nabla_X Y = \frac{1}{2}(\omega_2^{-1})^{\sharp}\big((\mathcal{L}_{jY}\omega_1)^{\sharp}X + j^*((\mathcal{L}_Y\omega_1)^{\sharp}X)\big).$$

This torsion-free connection on TS appears in Behrend and Fantechi's work on the Donaldson-Thomas invariants [2]. We will return to it in Corollary 5.7.

3.3. **Proofs of Theorems 3.13 and 3.14.**

Proof of Theorem 3.13. Let $\nabla : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ be the bilinear map as defined in Equation (3.8).

(1) We will prove that $\nabla I = \nabla J = \nabla K = 0$.

By the properties of Courant algebroid and almost hypercomplex conditions, we can verify that ∇ is indeed a hypercomplex connection.

Given that for all $U \in \Gamma(E), Y \in \Gamma(L_J)$,

$$\nabla_U Y = -I\frac{iJ+1}{2}(Y \circ IU - I(Y \circ U)) = -\frac{1-iJ}{2}I(Y \circ IU - I(Y \circ U)) \in \Gamma(L_J),$$

we have $\nabla_U JY = i\nabla_U Y = J\nabla_U Y$. Similarly, we have $\nabla_U J\xi = J\nabla_U \xi$, $\forall U \in \Gamma(E), \xi \in \Gamma(L_J^*)$. Therefore, $\nabla J = 0$.

A simple computation shows that, for all $U \in \Gamma(E), Y \in \Gamma(L_J)$,

$$\begin{aligned}
&I\nabla_U Y - \nabla_U IY \\
=&\frac{iJ+1}{2}(Y \circ IU - I(Y \circ U)) + \frac{iJ+1}{2}I(IY \circ IU - I(IY \circ U)) \\
=&\frac{iJ+1}{4}I\mathcal{N}(I,I)(Y,U) = 0.
\end{aligned}$$

Hence, we have $I\nabla_U Y = \nabla_U IY$. Similarly, we have $I\nabla_U \xi = \nabla_U I\xi, \forall U \in \Gamma(E), \xi \in \Gamma(L_J^*)$. Therefore, $\nabla I = 0$.

As $K = IJ$, we have $\nabla_U K = (\nabla_U I)J + I\nabla_U J = 0, \forall U \in \Gamma(E)$. Thus, $\nabla K = 0$.

(2) Next we will prove Equation (3.10). We claim that if (I,J,K) is an almost hypercomplex structure on a Courant algebroid $(E,\rho,\langle,\rangle,\circ)$, then, for all $U,V \in \Gamma(E)$,

$$T^{\nabla}(U,V) - \frac{1}{2}K\mathcal{N}(I,J)(U,V) = I\mathcal{D}\langle U,IV\rangle + J\mathcal{D}\langle U,JV\rangle + K\mathcal{D}\langle U,KV\rangle. \tag{3.11}$$

For all $U, V \in \Gamma(E)$, by the Courant algebroid properties and almost hypercomplex conditions, we have

$$\begin{aligned}
&T^{\nabla}(U,V) - \frac{1}{2}K\mathcal{N}(I,J)(U,V) \\
=&\nabla_U V - \nabla_V U - [\![U,V]\!] - \frac{1}{2}K\mathcal{N}(I,J)(U,V) \\
=&-\frac{1}{2}K(JV\circ IU - J(V\circ IU) - I(JV\circ U) + JI(V\circ U)) \\
&+\frac{1}{2}K(JU\circ IV - J(U\circ IV) - I(JU\circ V) + JI(U\circ V)) - \frac{1}{2}(U\circ V - V\circ U) \\
&-\frac{1}{2}K(IU\circ JV - I(U\circ JV) - J(IU\circ V) + JU\circ IV - J(U\circ IV) - I(JU\circ V)) \\
=&-\frac{1}{2}K(JV\circ IU - J(V\circ IU) - I(JV\circ U) + IU\circ JV - I(U\circ JV) - J(IU\circ V)) \\
=&-K(\mathcal{D}\langle IU, JV\rangle - I\mathcal{D}\langle U, JV\rangle - J\mathcal{D}\langle IU, V\rangle) \\
=&I\mathcal{D}\langle U, IV\rangle + J\mathcal{D}\langle U, JV\rangle + K\mathcal{D}\langle U, KV\rangle.
\end{aligned}$$

As $\mathcal{N}(I,J) = 0$, Equation (3.10) thus follows.

(3) We will now prove the uniqueness of the hypercomplex connection that satisfies Equations (3.9) and (3.10). Assume there exist two hypercomplex connections ∇^1 and ∇^2 that satisfy Equations (3.9) and (3.10). For all $U, V \in \Gamma(E)$, set $\Xi(U,V) = \nabla^1_U V - \nabla^2_U V$. It follows from Equation (3.9) that

$$\Xi(U, IV) = I\Xi(U,V), \quad \Xi(U, JV) = J\Xi(U,V), \quad \Xi(U, KV) = K\Xi(U,V),$$

and from Equation (3.10) that $\Xi(U,V) = \Xi(V,U)$. Therefore,

$$\begin{aligned}
&K\Xi(U,U) = IJ\Xi(U,U) = I\Xi(U,JU) = I\Xi(JU,U) = \Xi(JU,IU) \\
=&\Xi(IU,JU) = J\Xi(IU,U) = J\Xi(U,IU) = JI\Xi(U,U) = -K\Xi(U,U).
\end{aligned}$$

Hence, $\Xi(U,U) = 0$ for all $U \in \Gamma(E)$. Consequently,

$$\Xi(U,V) = \frac{1}{2}(\Xi(U+V,U+V) - \Xi(U,U) - \Xi(V,V)) = 0$$

for all $U, V \in \Gamma(E)$. Thus, the uniqueness of the hypercomplex connection satisfying Equations (3.9) and (3.10) is established. ☐

Theorem 3.14 follows from Theorem 3.13 and the next three lemmas.

Lemma 3.20. *Let (I,J,K) be an almost hypercomplex structure on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$. Assume that there exists a hypercomplex connection ∇ satisfying $\nabla I = \nabla J = \nabla K = 0$ and $T^{\nabla}(U,V) = I\mathcal{D}\langle U, IV\rangle + J\mathcal{D}\langle U, JV\rangle + K\mathcal{D}\langle U, KV\rangle$ for all $U, V \in \Gamma(E)$. Then $\mathcal{N}(I,J) = 0$.*

Proof. Assume that there exists a hypercomplex connection ∇ satisfying Equations (3.9) and (3.10). Equation (3.10) implies that

$$U\circ V = \nabla_U V - \nabla_V U + \mathcal{D}\langle U, V\rangle - (I\mathcal{D}\langle U, IV\rangle + J\mathcal{D}\langle U, JV\rangle + K\mathcal{D}\langle U, KV\rangle)$$

for all $U, V \in \Gamma(E)$. Therefore,

$$IU\circ JV = \nabla_{IU} JV - \nabla_{JV} IU + \mathcal{D}\langle IU, JV\rangle - (I\mathcal{D}\langle IU, IJV\rangle + J\mathcal{D}\langle IU, JJV\rangle + K\mathcal{D}\langle IU, KJV\rangle).$$

Since $\nabla I = \nabla J = 0$, it follows that

$$IU\circ JV = J\nabla_{IU} V - I\nabla_{JV} U - \mathcal{D}\langle U, KV\rangle - I\mathcal{D}\langle U, JV\rangle - J\mathcal{D}\langle U, IV\rangle + K\mathcal{D}\langle U, V\rangle.$$

Similarly, we have

$$
\begin{aligned}
JU\circ IV &= I\nabla_{JU}V - J\nabla_{IV}U + \mathcal{D}\langle U, KV\rangle - I\mathcal{D}\langle U, JV\rangle - J\mathcal{D}\langle U, IV\rangle - K\mathcal{D}\langle U, V\rangle;\\
U\circ JV &= J\nabla_{U}V - \nabla_{JV}U + \mathcal{D}\langle U, JV\rangle - I\mathcal{D}\langle U, KV\rangle + J\mathcal{D}\langle U, V\rangle + K\mathcal{D}\langle U, IV\rangle;\\
JU\circ V &= \nabla_{JU}V - J\nabla_{V}U - \mathcal{D}\langle U, JV\rangle - I\mathcal{D}\langle U, KV\rangle - J\mathcal{D}\langle U, V\rangle + K\mathcal{D}\langle U, IV\rangle;\\
U\circ IV &= I\nabla_{U}V - \nabla_{IV}U + \mathcal{D}\langle U, IV\rangle + I\mathcal{D}\langle U, V\rangle + J\mathcal{D}\langle U, KV\rangle - K\mathcal{D}\langle U, JV\rangle;\\
IU\circ V &= \nabla_{IU}V - I\nabla_{V}U - \mathcal{D}\langle U, IV\rangle - I\mathcal{D}\langle U, V\rangle + J\mathcal{D}\langle U, KV\rangle - K\mathcal{D}\langle U, JV\rangle.
\end{aligned}
$$

A simple computation shows that

$$\mathcal{N}(I,J)(U,V) = IU\circ JV - I(U\circ JV) - J(IU\circ V) + JU\circ IV - J(U\circ IV) - I(JU\circ V) = 0$$

for all $U, V \in \Gamma(E)$. Hence, $\mathcal{N}(I,J) = 0$. □

Lemma 3.21. *Let (I, J, K) be an almost hypercomplex structure on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$. If $\mathcal{N}(I,J) = 0$, then $\mathcal{N}(I,I) = \mathcal{N}(J,J) = 0$.*

Proof. Since for all $X, Y \in \Gamma(L_J)$,

$$
\begin{aligned}
&\mathcal{N}(I,J)(X,Y) = IX\circ JY - I(X\circ JY) - J(IX\circ Y) + JX\circ IY - J(X\circ IY) - I(JX\circ Y)\\
=&(i-J)(IX\circ Y + X\circ IY) - 2iI(X\circ Y),
\end{aligned}
$$

we have $X\circ Y = -\frac{1-iJ}{2}I(IX\circ Y + X\circ IY) \in \Gamma(L_J)$ from $N(I,J) = 0$. Thus, L_J is involutive, or equivalently $\mathcal{N}(J,J) = 0$. Similarly, $\mathcal{N}(I,I) = 0$. □

Lemma 3.22. *Let (I, J, K) be an almost hypercomplex structure on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$. If $\mathcal{N}(I,I) = \mathcal{N}(J,J) = 0$, then all six Nijenhuis tensors $\mathcal{N}(I,I)$, $\mathcal{N}(I,J)$, $\mathcal{N}(I,K)$, $\mathcal{N}(J,J)$, $\mathcal{N}(J,K)$, and $\mathcal{N}(K,K)$ vanish.*

Proof. First, we will prove $\mathcal{N}(I,J) = 0$ and $\mathcal{N}(K,K) = 0$ by checking that $\mathcal{N}(I,J)(X,Y) = \mathcal{N}(I,J)(\xi,\eta) = \mathcal{N}(I,J)(X,\xi) = 0$ and $\mathcal{N}(K,K)(X,Y) = \mathcal{N}(K,K)(\xi,\eta) = \mathcal{N}(K,K)(X,\xi) = 0$ for all $X, Y \in \Gamma(L_J), \xi, \eta \in \Gamma(L_J^*)$.

Let

$$
\begin{aligned}
P_1(X,Y) &= IX\circ IY - I\frac{1-iJ}{2}(X\circ IY) - I\frac{1-iJ}{2}(IX\circ Y),\\
P_2(X,Y) &= I\frac{1+iJ}{2}(X\circ IY) + I\frac{1+iJ}{2}(IX\circ Y) + X\circ Y.
\end{aligned}
$$

Then we have $P_1(X,Y) \in \Gamma(L_J^*)$ and $P_2(X,Y) \in \Gamma(L_J)$, since I swaps L_J and L_J^* and since both L_J, L_J^* are involutive. A simple computation shows that $\frac{1}{2}\mathcal{N}(I,I)(X,Y) = P_1(X,Y) - P_2(X,Y)$. Thus, we have $P_1(X,Y) = P_2(X,Y) = 0$. As a consequence, by a straightforward computation, we get

$$\mathcal{N}(K,K)(X,Y) = -2P_1(X,Y) - 2P_2(X,Y) = 0, \tag{3.12}$$

$$\mathcal{N}(I,J)(X,Y) = -2iIP_2(X,Y) = 0. \tag{3.13}$$

Similarly,

$$\mathcal{N}(I,J)(\xi,\eta) = \mathcal{N}(K,K)(\xi,\eta) = 0. \tag{3.14}$$

Moreover, we have

$$\mathcal{N}(I,J)(X,\xi) = (-i-J)(IX\circ\xi) + (i-J)(X\circ I\xi) = 0. \tag{3.15}$$

Now $\mathcal{N}(I,J) = 0$ follows from Equations (3.13), (3.14), and (3.15). On the other hand,

$$\frac{1}{2}\mathcal{N}(K,K)(X,\xi) = \frac{1}{2}\mathcal{N}(IJ,IJ)(X,\xi) = IX\circ I\xi + iIJ(X\circ I\xi) - iIJ(IX\circ\xi) - X\circ\xi.$$

As I swaps L_J and L_J^*, and as both L_J, L_J^* are involutive, we have

$$\frac{1}{2}\mathcal{N}(K,K)(X,\xi) = IX \circ I\xi - I(X \circ I\xi) - I(IX \circ \xi) - X \circ \xi = \frac{1}{2}\mathcal{N}(I,I)(X,\xi) = 0. \tag{3.16}$$

It follows from Equations (3.12), (3.14) and (3.16) that $\mathcal{N}(K,K) = 0$.

Similarly, $\mathcal{N}(I,K) = 0$, since $\mathcal{N}(I,I) = \mathcal{N}(K,K) = 0$. And, $\mathcal{N}(J,K) = 0$, since $\mathcal{N}(J,J) = \mathcal{N}(K,K) = 0$.

This completes the proof. □

4. Holomorphic symplectic structures on Courant algebroids

4.1. **Holomorphic symplectic structure on an arbitrary Courant algebroid.** Let $(E, \rho, \langle , \rangle, \circ)$ be a Courant algebroid endowed with a complex structure J. The nondegenerate pairing $\langle , \rangle$ induces a bijection between sections of $\otimes^2 E_{\mathbb{C}}^*$ and endomorphisms of $E_{\mathbb{C}}$, which associates an endomorphism $\Omega^\sharp$ of $E_{\mathbb{C}}$ with a section Ω of $\otimes^2 E_{\mathbb{C}}^*$:

$$\Omega(U,V) = \langle \Omega^\sharp U, V\rangle, \qquad \forall U,V \in \Gamma(E_{\mathbb{C}}).$$

The complex vector bundle $E_{\mathbb{C}}$ decomposes as the direct sum $L_J \oplus \overline{L}_J \cong L_J \oplus L_J^*$, where we identify $\overline{L}_J$ and L_J^*, therefore, an endomorphism $\Omega^\sharp$ of $E_{\mathbb{C}}$ skew-symmetric w.r.t. the pairing $\langle , \rangle$ corresponds to a section Ω of

$$\wedge^2 E_{\mathbb{C}}^* = \wedge^2 L_J \oplus (L_J \wedge L_J^*) \oplus \wedge^2 L_J^*. \tag{4.1}$$

In particular, whenever $\Omega^\sharp(L_J^*) = L_J$ and $\Omega^\sharp(L_J) = 0$, the components of Ω in $L_J \wedge L_J^*$ and $\wedge^2 L_J^*$ vanish. We can, therefore, consider Ω to be a section of $\wedge^2 L_J$.

Lemma 4.1. (1) *Let (I,J,K) be an almost hypercomplex structure on a Courant algebroid $(E,\rho,\langle , \rangle, \circ)$. The endomorphism $\Omega^\sharp = \frac{I+iK}{2}$ of $E_{\mathbb{C}}$ determines a section of $\wedge^2 L_J \subset \wedge^2 E_{\mathbb{C}}^*$ and satisfies*

$$\Omega^\sharp \overline{\Omega}^\sharp + \overline{\Omega}^\sharp \Omega^\sharp = -\operatorname{id}_{E_{\mathbb{C}}}. \tag{4.2}$$

(2) *Conversely, given an almost complex structure J on a Courant algebroid $(E,\rho,\langle , \rangle, \circ)$ and a section Ω of $\wedge^2 L_J \subset \wedge^2 E_{\mathbb{C}}^*$ satisfying $\Omega^\sharp \overline{\Omega}^\sharp + \overline{\Omega}^\sharp \Omega^\sharp = -\operatorname{id}_{E_{\mathbb{C}}}$, the triple $\big(I = \Omega^\sharp + \overline{\Omega}^\sharp,\ J,\ K = (-i)(\Omega^\sharp - \overline{\Omega}^\sharp)\big)$ is an almost hypercomplex structure on E.*

Proof. **(1)** The fact that the bilinear form Ω is skew-symmetric is a direct consequence of the skew-symmetry of I, K. Thus, Ω is a section of

$$\wedge^2 E_{\mathbb{C}}^* \cong \wedge^2 L_J \oplus (L_J \wedge L_J^*) \oplus \wedge^2 L_J^*.$$

Since $\Omega^\sharp = I\big(\frac{1+iJ}{2}\big)$ and I swaps L_J and L_J^*, we have $\Omega^\sharp(L_J^*) = L_J$, $\Omega^\sharp(L_J) = 0$, $\overline{\Omega}^\sharp(L_J) = L_J^*$, and $\overline{\Omega}^\sharp(L_J^*) = 0$. Therefore, $\Omega \in \Gamma(\wedge^2 L_J)$ and $\overline{\Omega} \in \Gamma(\wedge^2 L_J^*)$. Finally, we have

$$\Omega^\sharp \overline{\Omega}^\sharp + \overline{\Omega}^\sharp \Omega^\sharp = \Big(\frac{I+iK}{2}\Big)\Big(\frac{I-iK}{2}\Big) + \Big(\frac{I-iK}{2}\Big)\Big(\frac{I+iK}{2}\Big) = -1.$$

(2) As $\Omega \in \Gamma(\wedge^2 L_J)$, we have $(\Omega^\sharp)^2 = (\overline{\Omega}^\sharp)^2 = 0$. It follows that $I^2 = K^2 = \Omega^\sharp \overline{\Omega}^\sharp + \overline{\Omega}^\sharp \Omega^\sharp = -1$. For all $X \in \Gamma(L_J), \xi \in \Gamma(L_J^*)$, we have $\Omega^\sharp X = 0$, and $\overline{\Omega}^\sharp \xi = 0$; consequently, we have

$$IX = \overline{\Omega}^\sharp X,\ I\xi = \Omega^\sharp \xi,\ KX = i\overline{\Omega}^\sharp X,\ K\xi = -i\Omega^\sharp \xi.$$

Thus,

$$(IJ)(X+\xi) = iIX - iI\xi = i\overline{\Omega}^\sharp X - i\Omega^\sharp \xi = K(X+\xi).$$

Therefore, we have $K = IJ$. The orthogonality of I with respect to the pairing $\langle , \rangle$ follows from $I^2 = -1$ and the skew-symmetry of $I = \Omega^\sharp + \overline{\Omega}^\sharp$, while, the orthogonality of K follows from the identity $K = IJ$. □

Remark 4.2. Equation (4.2) means that $-\overline{\Omega}^\sharp : L_J \to L_J^*$ is the inverse map of $\Omega^\sharp : L_J^* \to L_J$. Thus, Equation (4.2) can be regarded as a nondegeneracy condition on Ω.

The following lemma, which can easily be verified, will be needed later on.

Lemma 4.3. *Assume that (I, J, K) is an almost hypercomplex structure on a Courant algebroid and set $\Omega^\sharp = \frac{1}{2}(I + iK)$. Then,*

(1) *$e \in L_J^*$ if and only if $\Omega^\sharp e = Ie = iKe$,*
(2) *$e \in L_J$ if and only if $\overline{\Omega}^\sharp e = Ie = -iKe$.*

We are now ready to introduce the following definition.

Definition 4.4. Let J be a complex structure on a Courant algebroid E. A *holomorphic symplectic structure on E with respect to J* is a section Ω of $\wedge^2 L_J$ satisfying

$$d_{L_J^*}\Omega = 0 \qquad \text{and} \qquad \Omega^\sharp\overline{\Omega}^\sharp + \overline{\Omega}^\sharp\Omega^\sharp = -\operatorname{id}_{E_{\mathbb{C}}}.$$

Given a hypercomplex triple (I, J, K), Lemma 3.3 implies that $\mathcal{N}_{I,J}, \mathcal{N}_{J,K} \in \Gamma(\wedge^3 E^*)$. We now extend $\mathcal{N}_{I,J}$ and $\mathcal{N}_{J,K}$ $\mathbb{C}$-linearly to 3-forms on $E_{\mathbb{C}}$.

Lemma 4.5. *Let (I, J, K) be a hypercomplex triple on a Courant algebroid $(E, \rho, \langle , \rangle, \circ)$, and let Ω be the section of $\wedge^2 L_J$ defined by $\Omega^\sharp = \frac{I+\sqrt{-1}K}{2}$. Then*

$$\frac{1}{4}\mathcal{N}_{I,J} = \frac{d_{L_J^*}\Omega - \overline{d_{L_J^*}\Omega}}{2i} \tag{4.3}$$

$$-\frac{1}{4}\mathcal{N}_{J,K} = \frac{d_{L_J^*}\Omega + \overline{d_{L_J^*}\Omega}}{2}. \tag{4.4}$$

Proof. First, we note that $\mathcal{N}_{I,J}$ is a section of $\wedge^3 L_J \oplus \wedge^3 L_J^*$ because $\langle \mathcal{N}(I,J)(X,Y), \xi\rangle = 0$ and $\langle \mathcal{N}(I,J)(\xi,\eta), X\rangle = 0$ for all $X, Y \in \Gamma(L_J)$ and $\xi, \eta \in \Gamma(L_J^*)$. Indeed, $\langle \mathcal{N}(I,J)(X,Y), \xi\rangle = 0$ follows from

$$\mathcal{N}(I,J)(X,Y) = (i - J)(IX \circ Y + X \circ IY) - 2iI(X \circ Y) \in \Gamma(L_J^*),$$

and $\langle \mathcal{N}(I,J)(\xi,\eta), X\rangle = 0$ follows from

$$\mathcal{N}(I,J)(\xi,\eta) = (-i - J)(I\xi \circ \eta + \xi \circ I\eta) + 2iI(\xi \circ \eta) \in \Gamma(L_J).$$

For all $\xi, \eta, \zeta \in \Gamma(L_J^*)$, we have

$$\begin{aligned}
&\langle \mathcal{N}(I,J)(\xi,\eta), \zeta\rangle = \langle (-i - J)(I\xi \circ \eta + \xi \circ I\eta) + 2iI(\xi \circ \eta), \zeta\rangle \\
&= -2i\langle I\xi \circ \eta, \zeta\rangle - 2i\langle \xi \circ I\eta, \zeta\rangle + 2i\langle I(\xi \circ \eta), \zeta\rangle = -2i\langle \Omega^\sharp\xi \circ \eta, \zeta\rangle - 2i\langle \xi \circ \Omega^\sharp\eta, \zeta\rangle + 2i\langle \Omega^\sharp(\xi \circ \eta), \zeta\rangle \\
&= -2i\langle 2\mathcal{D}\langle \Omega^\sharp\xi, \eta\rangle, \zeta\rangle + 2i\langle \eta \circ \Omega^\sharp\xi, \zeta\rangle - 2i\langle \xi \circ \Omega^\sharp\eta, \zeta\rangle + 2i\langle \Omega^\sharp(\xi \circ \eta), \zeta\rangle \\
&= -2i\rho(\zeta)\langle \Omega^\sharp\xi, \eta\rangle + 2i\rho(\eta)\langle \Omega^\sharp\xi, \zeta\rangle - 2i\langle \Omega^\sharp\xi, \eta \circ \zeta\rangle - 2i\rho(\xi)\langle \Omega^\sharp\eta, \zeta\rangle + 2i\langle \Omega^\sharp\eta, \xi \circ \zeta\rangle + 2i\langle \Omega^\sharp(\xi \circ \eta), \zeta\rangle \\
&= -2i\rho(\zeta)\Omega(\xi,\eta) + 2i\rho(\eta)\Omega(\xi,\zeta) - 2i\Omega(\xi, \eta \circ \zeta) - 2i\rho(\xi)\Omega(\eta,\zeta) + 2i\Omega(\eta, \xi \circ \zeta) + 2i\Omega(\xi \circ \eta, \zeta) \\
&= -2i(d_{L_J^*}\Omega)(\xi,\eta,\zeta).
\end{aligned}$$

Now Equation (4.3) holds, as $\overline{\mathcal{N}_{I,J}} = \mathcal{N}_{I,J}$ and $L_J^* = \overline{L_J}$.

The analogous relation

$$-\frac{1}{4}\mathcal{N}_{J,K} = \frac{d_{L_J^*}\Omega + \overline{d_{L_J^*}\Omega}}{2}$$

can be proved in the same way. □

By Lemma 4.1, Lemma 4.5 and Theorem 3.14, we have the following theorem.

Theorem 4.6. (1) *If (I, J, K) is a hypercomplex triple on a Courant algebroid E, then the section Ω of $\wedge^2 L_J$ defined by $\Omega^\sharp = \frac{I+\sqrt{-1}K}{2}$ is a holomorphic symplectic structure on E relative to the complex structure J.*

(2) *Let J be a complex structure on a Courant algebroid E, and let $\Omega \in \Gamma(\wedge^2 L_J)$ be a holomorphic symplectic structure on E relative to the complex structure J. Then the triple $\big(I = \Omega^\sharp + \overline{\Omega}^\sharp, J, K = -i(\Omega^\sharp - \overline{\Omega}^\sharp)\big)$ is a hypercomplex structure on E.*

The pair of eigenbundles (L_J, L_J^*) of a complex structure J on a Courant algebroid constitutes a Lie bialgebroid; therefore, $(\Gamma(\wedge^* L_J), [,], d_{L_J^*})$ is a differential graded Lie algebra, where $[,]$ denotes the Schouten bracket and $d_{L_J^*}$ denotes the Lie algebroid differential.

Theorem 4.7. *Let L_J and L_J^* denote the eigenbundles of a complex structure J on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$, and let Ω be a section of $\wedge^2 L_J \subset \wedge^2 E_{\mathbb{C}}$ such that $\Omega^\sharp\overline{\Omega}^\sharp + \overline{\Omega}^\sharp\Omega^\sharp = -1$. The following assertions are equivalent:*

1. $[\Omega, \Omega] = 0$, *where $[\cdot,\cdot]$ stands for the Schouten bracket on $\Gamma(\wedge^\bullet L_J)$,*
2. $d_{L_J^*}\Omega = 0$,
3. $d_{L_J^*}\Omega + \frac{1}{2}[\Omega, \Omega] = 0$.

Theorem 4.7 follows from Theorem 4.6, Theorem 3.14, Lemma 4.9, and Lemma 4.10.

To prove Lemma 4.9 and Lemma 4.10, we need the following lemma, which is an application of Theorem 6.1 and Equations (23) and (24) in [11] in the case of the Lie bialgebroid (L_J, L_J^*).

Lemma 4.8. *Let L_J and L_J^* denote the eigenbundles of a complex structure J on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$, and let Ω be a section of $\wedge^2 L_J$:*

1. *The subbundle $(1 + \Omega^\sharp)L_J^*$ of $E_{\mathbb{C}}$ is involutive if and only if $d_{L_J^*}\Omega + \frac{1}{2}[\Omega, \Omega] = 0$.*
2. *For all $\xi, \eta \in \Gamma(L_J^*)$, we have $(\frac{1}{2}[\Omega,\Omega])^\sharp(\xi,\eta) = \Omega^\sharp(\mathcal{L}_{\Omega^\sharp\xi}\eta - \mathcal{L}_{\Omega^\sharp\eta}\xi + d_{L_J}\langle\xi, \Omega^\sharp\eta\rangle) - [\Omega^\sharp\xi, \Omega^\sharp\eta]$.*

Lemma 4.9. *Given the same hypothesis as in Theorem 4.7, we have*

$$\frac{1}{2}[\Omega,\Omega](\xi,\eta,\zeta) = \overline{d_{L_J^*}\Omega}(\Omega^\sharp\xi, \Omega^\sharp\eta, \Omega^\sharp\zeta), \quad \forall \xi,\eta,\zeta \in \Gamma(L_J^*).$$

Proof. According to Lemma 4.8, for all $\xi, \eta, \zeta \in \Gamma(L_J^*)$, we have

$$\frac{1}{2}[\Omega,\Omega](\xi,\eta,\zeta) = -\langle \mathcal{L}_{\Omega^\sharp\xi}\eta - \mathcal{L}_{\Omega^\sharp\eta}\xi + d_{L_J}\langle\xi,\Omega^\sharp\eta\rangle, \Omega^\sharp\zeta\rangle - \langle[\Omega^\sharp\xi, \Omega^\sharp\eta], \zeta\rangle.$$

Set $X = \Omega^\sharp\xi$, $Y = \Omega^\sharp\eta$, and $Z = \Omega^\sharp\zeta$. Then we have $X, Y, Z \in \Gamma(L_J)$. And, by Lemma 4.3, we have $\xi = -\overline{\Omega}^\sharp X, \eta = -\overline{\Omega}^\sharp Y, \zeta = -\overline{\Omega}^\sharp Z$. Moreover,

$$\begin{aligned}
&\frac{1}{2}[\Omega,\Omega](\xi,\eta,\zeta) = \langle \mathcal{L}_X\overline{\Omega}^\sharp Y - \mathcal{L}_Y\overline{\Omega}^\sharp X + d_{L_J}\langle\overline{\Omega}^\sharp X, Y\rangle, Z\rangle + \langle[X,Y], \overline{\Omega}^\sharp Z\rangle \\
=&\rho(X)\overline{\Omega}(Y,Z) - \overline{\Omega}(Y,[X,Z]) - \rho(Y)\overline{\Omega}(X,Z) + \overline{\Omega}(X,[Y,Z]) + \rho(Z)\overline{\Omega}(X,Y) - \overline{\Omega}([X,Y],Z) \\
=&d_{L_J}\overline{\Omega}(X,Y,Z) = \overline{d_{\overline{L}_J}\Omega}(X,Y,Z) \\
=&\overline{d_{L_J^*}\Omega}(\Omega^\sharp\xi, \Omega^\sharp\eta, \Omega^\sharp\zeta).
\end{aligned}$$

This completes the proof. □

Lemma 4.10. *Given the same hypothesis as in Theorem 4.7, set $K = (-i)(\Omega^\sharp - \overline{\Omega}^\sharp)$. We have*

$$\mathcal{N}(K,K) = 0 \qquad \textit{if and only if} \qquad d_{L_J^*}\Omega + \tfrac{1}{2}[\Omega,\Omega] = 0.$$

Proof. By assumption, we have $\mathcal{N}(J,J)=0$. By Lemma 4.8, the subbundle

$$(1+\Omega^\sharp)L_J^* = \{\xi+\Omega^\sharp\xi \in E_\mathbb{C} | \xi \in L_J^*\}$$

of $E_\mathbb{C}$ is involutive if and only if $d_{L_J^*}\Omega + \frac{1}{2}[\Omega,\Omega]=0$. Since $L_K^* = \frac{1+I}{\sqrt{2}}L_J^*$ and $I\xi = \Omega^\sharp\xi$ for all $\xi \in L_J^*$ (see Lemma 4.3), we have $L_K^* = (1+\Omega^\sharp)L_J^*$. Since the involutivity of L_K^* is equivalent to the vanishing of the Nijenhuis concomitant $\mathcal{N}(K,K)$, the result follows. □

A complex structure J on Courant algebroid E is equivalent to a decomposition of $E_\mathbb{C}$ as a direct sum $L \oplus \overline{L}$ of complex conjugate, maximal isotropic, involutive subbundles, namely the eigenbundles of J relative to the eigenvalues $\pm\sqrt{-1}$ (see [5]).

Lemma 4.11. *Let L_J, L_J^* be the eigenbundles of a complex structure J on a Courant algebroid E. Given $\Omega \in \Gamma(\wedge^2 L_J)$, the subbundle $(1+\Omega^\sharp)L_J^*$ is the eigenbundle of a complex structure on E if and only if $d_{L_J^*}\Omega + \frac{1}{2}[\Omega,\Omega]=0$ and $\overline{\Omega}^\sharp\Omega^\sharp - \mathrm{id}_{L_J^*}$ is an invertible endomorphism of L_J^*.*

A holomorphic symplectic 2-form on a Courant algebroid can be interpreted in the light of Lemma 4.11 as a deformation of the given complex structure into a 2-dimensional sphere of complex structures (see also Proposition 3.5).

Proposition 4.12. *Let J be a complex structure on a Courant algebroid E and let $\Omega \in \Gamma(\wedge^2 L_J)$ be a holomorphic symplectic structure on E relative to J. Set $I = \Omega^\sharp + \overline{\Omega}^\sharp$ and $K = -\sqrt{-1}(\Omega^\sharp - \overline{\Omega}^\sharp)$. Then, for any $a,b\in\mathbb{R}$, the endomorphism $\frac{1-a^2-b^2}{1+a^2+b^2}J + \frac{2a}{1+a^2+b^2}K + \frac{2b}{1+a^2+b^2}I$ is a complex structure on E and $(1+(a+b\sqrt{-1})\Omega^\sharp)L_J^*$ is the subbundle associated with its eigenvalue $-\sqrt{-1}$.*

Note that the map

$$S^2 \ni \left(\frac{1-a^2-b^2}{1+a^2+b^2}, \frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}\right) \mapsto a+b\sqrt{-1} \in \mathbb{C}\cup\{\infty\}$$

is the stereographic projection.

Proof. Set $I = \Omega^\sharp + \overline{\Omega}^\sharp$ and $K = \sqrt{-1}(\overline{\Omega}^\sharp - \Omega^\sharp)$. Then (I,J,K) is a hypercomplex structure on the Courant algebroid E, and $\frac{1-a^2-b^2}{1+a^2+b^2}J + \frac{2a}{1+a^2+b^2}K + \frac{2b}{1+a^2+b^2}I$ is a complex structure on E by Proposition 3.5. Since

$$\left(\frac{1+aI-bK}{\sqrt{1+a^2+b^2}}\right) J \left(\frac{1-aI+bK}{\sqrt{1+a^2+b^2}}\right) = \frac{1-a^2-b^2}{1+a^2+b^2}J + \frac{2a}{1+a^2+b^2}K + \frac{2b}{1+a^2+b^2}I, \tag{4.5}$$

and

$$\frac{1-aI+bK}{\sqrt{1+a^2+b^2}} = \left(\frac{1+aI-bK}{\sqrt{1+a^2+b^2}}\right)^{-1},$$

it follows that J and $\frac{1-a^2-b^2}{1+a^2+b^2}J + \frac{2a}{1+a^2+b^2}K + \frac{2b}{1+a^2+b^2}I$ are conjugate endomorphisms of the Courant algebroid E and that $\frac{1+aI-bK}{\sqrt{1+a^2+b^2}}$ maps L_J^* to the $-\sqrt{-1}$ eigenbundle of $\frac{1-a^2-b^2}{1+a^2+b^2}J + \frac{2a}{1+a^2+b^2}K + \frac{2b}{1+a^2+b^2}I$.

On the other hand, by Lemma 4.3, we have

$$(1+(a+b\sqrt{-1})\Omega^\sharp)\xi = \xi + a\Omega^\sharp\xi + b\sqrt{-1}\Omega^\sharp\xi = (1+aI-bK)\xi$$

for any element $\xi \in L_J^*$.

Therefore, $(1+(a+b\sqrt{-1})\Omega^\sharp)L_J^*$ is the eigenbundle of $\frac{1-a^2-b^2}{1+a^2+b^2}J + \frac{2a}{1+a^2+b^2}K + \frac{2b}{1+a^2+b^2}I$ associated with the eigenvalue $-\sqrt{-1}$. □

4.2. **Holomorphic symplectic structures on $T\oplus T^*$.** Consider the complex structure
$$J=\begin{pmatrix} j & 0\\ 0 & -j^*\end{pmatrix}$$
on the Courant algebroid $T\oplus T^*$ associated with a complex manifold with complex structure j. Its eigenbundles are $L_J=T^{1,0}\oplus(T^{0,1})^*$ and $L_J^*=T^{0,1}\oplus(T^{1,0})^*$. Assume that $\Omega\in\Gamma(\wedge^2 L_J)$ is a holomorphic symplectic structure on $T\oplus T^*$ relatively to J. Since
$$\wedge^2 L_J=\wedge^2 T^{1,0}\oplus(T^{1,0}\wedge(T^{0,1})^*)\oplus\wedge^2(T^{0,1})^*,$$
Ω decomposes as $\Omega=\pi+\theta+\omega$ where $\pi\in\Gamma(\wedge^2T^{1,0})$, $\theta\in\Gamma(T^{1,0}\wedge(T^{0,1})^*)$ and $\omega\in\Gamma(\wedge^2(T^{0,1})^*)$. Then $\Omega^\sharp\overline{\Omega}^\sharp+\overline{\Omega}^\sharp\Omega^\sharp=-1$ is equivalent to the following equations:
$$\begin{aligned}\pi^\sharp\overline{\theta}^\sharp+\theta^\sharp\overline{\pi}^\sharp&=0; & \pi^\sharp\overline{\omega}^\sharp+\theta^\sharp\overline{\theta}^\sharp&=-1;\\ \omega^\sharp\overline{\pi}^\sharp+\theta^\sharp\overline{\theta}^\sharp&=-1; & \omega^\sharp\overline{\theta}^\sharp+\theta^\sharp\overline{\omega}^\sharp&=0.\end{aligned}$$
The condition $d_{L_J^*}\Omega=0$ is equivalent to
$$\overline{\partial}\pi=0;\qquad \overline{\partial}\theta=0;\qquad \overline{\partial}\omega=0,$$
as $d_{L_J^*}=\bar\partial$ in the present context. However, $[\Omega,\Omega]=0$ is equivalent to
$$\begin{aligned}[]{}[\pi,\pi]&=0; & [\pi,\theta]&=0;\\ 2[\pi,\omega]+[\theta,\theta]&=0; & [\theta,\omega]&=0.\end{aligned}$$
As a consequence, π is necessarily a holomorphic Poisson structure on M.

In conclusion, we have

Proposition 4.13. *Given a complex manifold with complex structure j, if π, θ, and ω are sections of $\wedge^2T^{1,0}$, $T^{1,0}\wedge(T^{0,1})^*$, and $\wedge^2(T^{0,1})^*$ respectively, and satisfy the relations*
$$\overline{\partial}\pi=0;\quad \overline{\partial}\theta=0;\quad \overline{\partial}\omega=0,$$
and
$$\begin{aligned}\pi^\sharp\overline{\theta}^\sharp+\theta^\sharp\overline{\pi}^\sharp&=0; & \pi^\sharp\overline{\omega}^\sharp+\theta^\sharp\overline{\theta}^\sharp&=-1;\\ \omega^\sharp\overline{\pi}^\sharp+\theta^\sharp\overline{\theta}^\sharp&=-1; & \omega^\sharp\overline{\theta}^\sharp+\theta^\sharp\overline{\omega}^\sharp&=0,\end{aligned}$$
then $\Omega=\pi+\theta+\omega$ is a holomorphic symplectic structure on $T\oplus T^*$ relative to $J=\left(\begin{smallmatrix} j & 0\\ 0 & -j^*\end{smallmatrix}\right)$, and we have
$$\begin{aligned}[]{}[\pi,\pi]&=0; & [\pi,\theta]&=0;\\ 2[\pi,\omega]+[\theta,\theta]&=0; & [\theta,\omega]&=0.\end{aligned}$$
In particular, π is necessarily a holomorphic Poisson structure.

An *extended Poisson structure* on a complex manifold (with complex structure j) is an element $\Omega=\pi+\theta+\omega$ of $\wedge^2\big(T^{1,0}\oplus(T^{0,1})^*\big)$ satisfying $\bar\partial\Omega+\frac{1}{2}[\Omega,\Omega]=0$ (see [3]). If Ω satisfies the additional algebraic condition $\Omega^\sharp\overline{\Omega}^\sharp+\overline{\Omega}^\sharp\Omega^\sharp=-1$, then Ω is holomorphic symplectic w.r.t. $J=\left(\begin{smallmatrix} j & 0\\ 0 & -j^*\end{smallmatrix}\right)$ by Theorem 4.7.

This motivates the following definition.

Definition 4.14. An *extended symplectic structure* on a complex manifold$(M;j)$ is a holomorphic symplectic structure on the standard Courant algebroid $T\oplus T^*$ relative to the complex structure $J=\left(\begin{smallmatrix} j & 0\\ 0 & -j^*\end{smallmatrix}\right)$.

Example 4.15. If $\omega = \pi = 0$, the equations in Proposition 4.13 become

$$\bar{\partial}\theta = 0 \quad \text{and} \quad \theta^\sharp \overline{\theta}^\sharp = -1.$$

Setting $i = \theta^\sharp + \overline{\theta}^\sharp$ and $k = -\sqrt{-1}(\theta^\sharp - \overline{\theta}^\sharp)$, we recover a (classical) hypercomplex triple (i, j, k) on the manifold (see Example 3.8).

Example 4.16. If $\theta = 0$, the equations in Proposition 4.13 become

$$\pi = -(\overline{\omega})^{-1}, \quad \bar{\partial}\omega = 0, \quad \bar{\partial}\pi = 0, \quad [\pi, \pi] = 0 \quad \text{and} \quad [\omega, \pi] = 0.$$

Therefore, $\Omega = \omega + \pi$ is a holomorphic symplectic structure on $T \oplus T^*$ if and only if $\overline{\omega} = -\pi^{-1}$ is a holomorphic symplectic 2-form. Thus, we recover a holomorphic symplectic manifold (see Example 3.9).

4.3. **Hyper-Poisson structure.**

Definition 4.17 ([17])**.** Let (i, j, k) be a hypercomplex triple on a manifold M, and let $\omega_1, \omega_2, \omega_3$ be three 2-forms on M. If $\omega_2 + \sqrt{-1}\omega_3$ is a holomorphic symplectic 2-form with respect to the complex structure i, $\omega_3 + \sqrt{-1}\omega_1$ a holomorphic symplectic 2-form with respect to the complex structure j, and $\omega_1 + \sqrt{-1}\omega_2$ a holomorphic symplectic 2-form with respect to the complex structure k, then $(\omega_1, \omega_2, \omega_3)$ is a *hyper-symplectic structure* on M with respect to (i, j, k).

Remark 4.18. Definition 4.17 is invariant under cyclic permutations of i, j, k and $\omega_1, \omega_2, \omega_3$.

The following definition is a natural generalization of hyper-symplectic structures in the Poisson context.

Definition 4.19. Let (i, j, k) be a hypercomplex triple on a manifold M, and let π_1, π_2, and π_3 be three bivector fields on M. If $\pi_2 - \sqrt{-1}\pi_3$ is a holomorphic Poisson tensor with respect to the complex structure i, $\pi_3 - \sqrt{-1}\pi_1$ a holomorphic Poisson tensor with respect to the complex structure j, and $\pi_1 - \sqrt{-1}\pi_2$ a holomorphic Poisson tensor with respect to the complex structure k, then (π_1, π_2, π_3) is a *hyper-Poisson structure* on M with respect to (i, j, k).

Theorem 4.20. *The following assertions are equivalent to each other:*

(1) *The triple (i, j, k) is a hypercomplex structure on M and (π_1, π_2, π_3) is a hyper-Poisson structure with respect to (i, j, k) satisfying $\pi_2^\sharp = -i\pi_3^\sharp = -\pi_3^\sharp i^*$.*
(2) *The triple $I = \begin{pmatrix} i & \pi_3 \\ 0 & -i^* \end{pmatrix}$, $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$, $K = \begin{pmatrix} k & -\pi_1 \\ 0 & -k^* \end{pmatrix}$ is a hypercomplex structure on the standard Courant algebroid $T_M \oplus T_M^*$.*
(3) *Having defined $\theta \in \Gamma(T^{1,0} \wedge (T^{0,1})^*)$ (relatively to j) by $\theta^\sharp = \frac{1}{2}(i + \sqrt{-1}k)$ and $\pi \in \Gamma(\wedge^2 T^{1,0})$ (relatively to j) by $\pi = \frac{1}{2}(\pi_3 - \sqrt{-1}\pi_1)$, their sum $\Omega = \theta + \pi$ is a holomorphic symplectic structure on the standard Courant algebroid $T_M \oplus T_M^*$ relative to the complex structure $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$.*

In order to prove Theorem 4.20, we will make use of the following lemma or more precisely a corollary of it.

Lemma 4.21 ([10])**.** *Let M be a complex manifold (with complex structure j). If π_λ and π_μ are two real bivector fields on M such that $\pi_\lambda + \sqrt{-1}\pi_\mu$ is a holomorphic Poisson bivector field, then $\pi_\mu^\sharp = -j\pi_\lambda^\sharp = -\pi_\lambda^\sharp j^*$ and*

$$[\![\pi_\lambda, \pi_\lambda]\!] = [\![\pi_\mu, \pi_\mu]\!] = [\![\pi_\lambda, \pi_\mu]\!] = 0.$$

Corollary 4.22. *Given a hyper-Poisson structure (π_1, π_2, π_3) on a manifold M with respect to a hypercomplex triple (i, j, k), we have*

(1) *$[\![\pi_\alpha, \pi_\beta]\!] = 0$, for any $\alpha, \beta \in \{1, 2, 3\}$;*

(2) $\pi_3^\sharp = i\pi_2^\sharp = \pi_2^\sharp i^*$, $\pi_1^\sharp = j\pi_3^\sharp = \pi_3^\sharp j^*$, $\pi_2^\sharp = k\pi_1^\sharp = \pi_1^\sharp k^*$;
(3) $i\pi_1^\sharp = -\pi_1^\sharp i^* = j\pi_2^\sharp = -\pi_2^\sharp j^* = k\pi_3^\sharp = -\pi_3^\sharp k^*$.

Therefore, if one among π_1, π_2, and π_3 is invertible, then so are the other two. In this case, $(\pi_1^{-1}, \pi_2^{-1}, \pi_3^{-1})$ is a hyper-symplectic structure on M and $(i\pi_1^\sharp)^{-1}$ defines a pseudo-metric g on M. In particular, if g is positive definite, we obtain a hyper-Kähler structure on M.

Proof of Theorem 4.20. (1) $\Rightarrow$ (2) Assume that (π_1, π_2, π_3) is a hyper-Poisson structure on M with respect to a hypercomplex triple (i, j, k). As $\pi_3 + \sqrt{-1}\pi_2 = \sqrt{-1}(\pi_2 - \sqrt{-1}\pi_3)$ is a holomorphic Poisson structure with respect to the complex structure i, by Lemma 2.9, $I = \begin{pmatrix} i & \pi_3 \\ 0 & -i^* \end{pmatrix}$ is a complex structure on the standard Courant algebroid $TM \oplus T^*M$ and $\pi_2^\sharp = -i\pi_3^\sharp = -\pi_3^\sharp i^*$. Similarly, $K = \begin{pmatrix} k & -\pi_1 \\ 0 & -k^* \end{pmatrix}$ is a complex structure on $TM \oplus T^*M$. It follows from Corollary 4.22 that (I, J, K) is an almost hypercomplex structure on $TM \oplus T^*M$, and from Theorem 3.14 that (I, J, K) is a hypercomplex structure on $TM \oplus T^*M$.

(2) $\Leftrightarrow$ (3) The equivalence of (2) and (3) is a direct consequence of Theorem 4.6 and Proposition 4.13.

(2) $\Rightarrow$ (1) Given a hypercomplex structure $I = \begin{pmatrix} i & \pi_3 \\ 0 & -i^* \end{pmatrix}, J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}, K = \begin{pmatrix} k & -\pi_1 \\ 0 & -k^* \end{pmatrix}$ on the standard Courant algebroid $E = TM \oplus T^*M$, let $\theta \in \Gamma(T^{1,0} \wedge (T^{0,1})^*)$ and $\pi \in \Gamma(\wedge^2 T^{1,0})$ be defined by $\theta^\sharp = \frac{1}{2}(i + \sqrt{-1}k)$ and $\pi = \frac{1}{2}(\pi_3 - \sqrt{-1}\pi_1)$. Then $\Omega = \theta + \pi$ is a holomorphic symplectic form on E with respect to $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$, and, by Proposition 4.13, we have

$$\bar\partial\theta = \bar\partial\pi = 0, \quad [\pi, \pi] = 0, \quad [\pi, \theta] = 0, \quad [\theta, \theta] = 0, \quad \theta^\sharp\overline{\theta}^\sharp = -1, \quad \pi^\sharp\overline{\theta}^\sharp + \theta^\sharp\overline{\pi}^\sharp = 0.$$

Thus, (i, j, k) is a hypercomplex structure on M by our discussion in Example 4.15, and $\pi_3 - \sqrt{-1}\pi_1 = 2\pi$ is a holomorphic Poisson structure with respect to j. As $I = \begin{pmatrix} i & \pi_3 \\ 0 & -i^* \end{pmatrix}$ is a complex structure on $E = TM \oplus T^*M$, by Lemma 2.9, $\pi_2 - \sqrt{-1}\pi_3 = -\sqrt{-1}(\pi_3 + \sqrt{-1}\pi_2)$ is a holomorphic Poisson structure with respect to the complex structure i, where π_2 is defined by $\pi_2^\sharp = -i\pi_3^\sharp = -\pi_3 i^*$. And, $\pi_1 - \sqrt{-1}\pi_2$ is a holomorphic Poisson structure with respect to k, since $K = \begin{pmatrix} k & -\pi_1 \\ 0 & k^* \end{pmatrix}$ is a complex structure on $E = TM \oplus T^*M$ and $\pi_2^\sharp = -i\pi_3^\sharp = ij\pi_1^\sharp = k\pi_1^\sharp = \pi_1^\sharp k^*$. $\square$

Corollary 4.23. *If (π_1, π_2, π_3) is a hyper-Poisson structure on a smooth manifold M with respect to a hypercomplex structure (i, j, k), then π_1, π_2, and π_3 have the same symplectic leaves and each leaf is a hyper-symplectic manifold.*

Proof. As $\pi_2^\sharp = -\pi_3^\sharp i^*$ according to Theorem 4.20, it follows that $\pi_2^\sharp(T^*M) = \pi_3^\sharp(T^*M)$. Therefore, π_2 and π_3 have the same symplectic foliations. Moreover, since $\pi_2^\sharp = -i\pi_3^\sharp$, such symplectic foliations are stable under i. Hence, each symplectic leaf of π_2 (and π_3) is a complex submanifold with respect to i. As the rule of π_1, π_2, π_3 is symmetric, the conclusion follows. $\square$

5. Hypercomplex connection and Lagrangian Lie subalgebroid

5.1. A basic study of hypercomplex connection.

Lemma 5.1. *Let (I, J, K) be a hypercomplex structure on a Courant algebroid $(E, \rho, \langle,\rangle, \circ)$ and let $\Omega \in \wedge^2 L_J$ denote the corresponding holomorphic symplectic form relative to J defined by $\Omega^\sharp = \frac{I+iK}{2}$. For any $X, Y, Z \in \Gamma(L_J)$ and $\xi, \eta, \zeta \in \Gamma(L_J^*)$, we have*

(1) $\nabla_X\xi = \imath_X d_{L_J}(\xi) \in \Gamma(L_J^*)$ *and* $\nabla_\xi X = \imath_\xi d_{L_J^*}(X) \in \Gamma(L_J)$,
(2) $\nabla_X Y = -\Omega^\sharp(\imath_X \mathcal{L}_Y \overline{\Omega}) \in \Gamma(L_J)$ *and* $\nabla_\xi\eta = -\overline{\Omega}^\sharp(\imath_\xi \mathcal{L}_\eta \Omega) \in \Gamma(L_J^*)$,
(3) $R^\nabla(X, Y)Z = 0$ *and* $R^\nabla(\xi, \eta)\zeta = 0$,

where ∇ stands for the hypercomplex connection defined in Equation (3.8) *and R^∇ for its curvature.*

Proof. **(1)** For all $X,Y\in\Gamma(L_J),\xi\in\Gamma(L_J^*)$, by Equation (3.8), we have

$$\langle\nabla_X\xi,Y\rangle=\langle-\frac{1+iJ}{2}(\xi\circ X)+\frac{K}{2}(i+J)(\xi\circ IX),Y\rangle=-\langle\xi\circ X,Y\rangle,$$

as I swaps L_J and L_J^*, and as L_J^* is involutive. Thus,

$$\begin{aligned}\langle\nabla_X\xi,Y\rangle&=\langle-\xi\circ X,Y\rangle=\langle-2\mathcal{D}\langle\xi,X\rangle+X\circ\xi,Y\rangle\\&=-\rho(Y)\langle\xi,X\rangle+\rho(X)\langle\xi,Y\rangle-\langle\xi,X\circ Y\rangle=d_{L_J}(\xi)(X,Y),\end{aligned}$$

Now $\nabla_X\xi=\imath_Xd_{L_J}(\xi)$, as $\nabla J=0$ implies that $\nabla_X\xi\in L_J^*$.

Similarly, we have $\nabla_\xi X=\imath_\xi d_{L_J^*}(X)$.

(2) First, $\nabla J=0$ implies that $\nabla_\xi\eta\in\Gamma(L_J^*)$, $\forall\xi,\eta\in\Gamma(L_J^*)$. By Equation (3.8), we have

$$\nabla_\xi\eta=\frac{-I+iK}{2}(\eta\circ I\xi)-\eta\circ\xi=-\overline{\Omega}^\sharp(\eta\circ\Omega^\sharp\xi)-\eta\circ\xi.$$

As $\Omega^\sharp:L_J^*\to L_J$ is the inverse of $-\overline{\Omega}^\sharp:L_J\to L_J^*$, for all $\xi,\eta,\zeta\in\Gamma(L_J^*)$, we have

$$\Omega(\nabla_\xi\eta,\zeta)=\langle\eta\circ\Omega^\sharp\xi,\zeta\rangle-\Omega(\eta\circ\xi,\zeta)=(\mathcal{L}_\eta\Omega)(\xi,\zeta).$$

Therefore,

$$(\nabla_\xi\eta)\lrcorner\Omega=\xi\lrcorner(\mathcal{L}_\eta\Omega),\tag{5.1}$$

i.e. $\Omega^\sharp(\nabla_\xi\eta)=\imath_\xi(\mathcal{L}_\eta\Omega)$. From Remark 4.2, $\nabla_\xi\eta=-\overline{\Omega}^\sharp(\imath_\xi\mathcal{L}_\eta\Omega)$ follows.

Similarly, we have $\nabla_XY=-\Omega^\sharp(\imath_X\mathcal{L}_Y\overline{\Omega})\in\Gamma(L_J)$ for all $X,Y\in\Gamma(L_J)$.

(3) For all $\xi,\eta,\zeta\in\Gamma(L_J^*)$, by Equation (5.1), we have

$$\begin{aligned}R^\nabla(\xi,\eta)\zeta\lrcorner\Omega=&(\nabla_\xi\nabla_\eta\zeta-\nabla_\eta\nabla_\xi\zeta-\nabla_{[\![\xi,\eta]\!]}\zeta)\lrcorner\Omega\\=&\xi\lrcorner d_{L_J^*}(\nabla_\eta\zeta\lrcorner\Omega)-\eta\lrcorner d_{L_J^*}(\nabla_\xi\zeta\lrcorner\Omega)-[\![\xi,\eta]\!]\lrcorner d_{L_J^*}(\zeta\lrcorner\Omega)\\=&\xi\lrcorner d_{L_J^*}(\eta\lrcorner d_{L_J^*}(\zeta\lrcorner\Omega))-\eta\lrcorner d_{L_J^*}(\xi\lrcorner d_{L_J^*}(\zeta\lrcorner\Omega))-[\![\xi,\eta]\!]\lrcorner d_{L_J^*}(\zeta\lrcorner\Omega)\\=&\xi\lrcorner\mathcal{L}_\eta d_{L_J^*}(\zeta\lrcorner\Omega)-\eta\lrcorner\mathcal{L}_\xi d_{L_J^*}(\zeta\lrcorner\Omega)-[\![\xi,\eta]\!]\lrcorner d_{L_J^*}(\zeta\lrcorner\Omega)\\=&(\xi\lrcorner\mathcal{L}_\eta-\eta\lrcorner\mathcal{L}_\xi-[\![\xi,\eta]\!]\lrcorner)d_{L_J^*}(\zeta\lrcorner\Omega)\\=&0.\end{aligned}$$

Thus, $R^\nabla(\xi,\eta)\zeta=0$, $\forall\xi,\eta,\zeta\in\Gamma(L_J^*)$.

Similarly, $R^\nabla(X,Y)Z=0$, $\forall X,Y,Z\in\Gamma(L_J)$. ☐

Proposition 5.2. *Let (I,J,K) be a hypercomplex structure on a Courant algebroid $(E,\rho,\langle,\rangle,\circ)$, let ∇ be the associated hypercomplex connection given in Equation (3.8), and let $\Omega\in\wedge^2L_J$ be the symplectic form, holomorphic with respect to J, defined by $\Omega^\sharp=\frac{I+iK}{2}$. The following assertions are equivalent to each other for any $V\in\Gamma(E)$:*

(1) $\nabla V=0$;
(2) $d_{L_I}(V+iIV)=0$, $d_{L_J}(V+iJV)=0$, *and* $d_{L_K}(V+iKV)=0$;
(3) $d_{L_J}(V+iJV)=0$ *and* $\mathcal{L}_{V+iJV}\Omega=0$.

(Here we consider $V+iIV$ as a section of $\Gamma(\overline{L_I})$, $V+iJV$ as a section of $\Gamma(\overline{L_J})$, and $V+iKV$ as a section of $\Gamma(\overline{L_K})$.)

Proof. Set $\xi=V+iJV$, then $\xi\in\Gamma(L_J^*)$.

(1)⇒(2) Given $\nabla V=0$, we have $\nabla\xi=0$, as $\nabla J=0$. By Lemma 5.1, $\imath_Xd_{L_J}(\xi)=\nabla_X\xi=0$, for all $X\in\Gamma(L_J)$. As a consequence, $d_{L_J}(V+iJV)=d_{L_J}(\xi)=0$. Similarly, we have $d_{L_I}(V+iIV)=0$ and $d_{L_K}(V+iKV)=0$.

(2)⇒(1) For any $U\in\Gamma(E)$, as $U-iJU$ is a section of L_J, by Lemma 5.1, we have

$$\nabla_{U-iJU}(V+iJV)=\imath_{(U-iJU)}d_{L_J}(\xi)=0.$$

Since

$$\nabla_{U-iJU}(V+iJV)=(\nabla_U V+\nabla_{JU}JV)+i(\nabla_U JV-\nabla_{JU}V),$$

and since ∇ defined by Equation (3.8) is a real connection, we have $\nabla_U JV=\nabla_{JU}V$. Hence, by $\nabla J=0$, we have

$$\nabla_{JU}V=\nabla_U JV=J\nabla_U V. \tag{5.2}$$

Similarly,

$$\nabla_{IU}V=I\nabla_U V,\ \nabla_{KU}V=K\nabla_U V. \tag{5.3}$$

From Equations (5.2) and (5.3) and identity $\nabla I=\nabla J=\nabla K=0$, we have

$$K\nabla_U V=\nabla_U KV=\nabla_U IJV=I\nabla_U(JV)=\nabla_{IU}JV=J\nabla_{IU}V=JI\nabla_U V=-K\nabla_U V. \tag{5.4}$$

Therefore, $\nabla_U V=0$ for all $U\in\Gamma(E)$, i.e. $\nabla V=0$.

(1)$\Rightarrow$(3) As $\nabla V=0$ and $\nabla J=0$, we have $\nabla\xi=0$. It follows from Lemma 5.1 that $\nabla_X\xi=\imath_X d_{L_J}(\xi)$ and $\nabla_\eta\xi=-\overline{\Omega}^\sharp(\imath_\eta\mathcal{L}_\xi\Omega)$ for all $X\in\Gamma(L_J)$ and $\eta\in\Gamma(L_J^*)$. Thus, $d_{L_J}(\xi)=0$ and $\mathcal{L}_\xi\Omega=0$.

(3)$\Rightarrow$(1) Since $d_{L_J}(\xi)=0$ and $\mathcal{L}_\xi\Omega=0$, by Lemma 5.1, we have $\nabla\xi=0$. As ∇ is a real connection, we have $\nabla V=0$ from $\nabla(V+iJV)=\nabla\xi=0$. ☐

Remark 5.3. In Proposition 5.2, (2) can be understood as the requirement that the vector field V be the common real component of three complex vector fields, which are holomorphic with respect to the complex structures I, J, and K respectively, whereas (3) can be understood as the requirement that V be the real component of a holomorphic symplectic vector field.

5.2. **The induced connections on Lagrangian Lie subalgebroids.**

Definition 5.4. Let $(E,\rho,\langle,\rangle,\circ)$ be a Courant algebroid endowed with a complex structure J. Assume that $\Omega\in\Gamma(\wedge^2 L_J)$ is a holomorphic symplectic structure on E relative to the complex structure J. A Lie subalgebroid of L_J^* is said to be *Lagrangian* if it is maximal isotropic with respect to Ω.

Theorem 5.5. *Let $(E,\rho,\langle,\rangle,\circ)$ be a Courant algebroid endowed with a complex structure J. Assume that $\Omega\in\Gamma(\wedge^2 L_J)$ is holomorphic symplectic with respect to J, and set $I=\Omega^\sharp+\overline{\Omega}^\sharp$ and $K=-i(\Omega^\sharp-\overline{\Omega}^\sharp)$. Given a Lie subalgebroid L of L_J^* Lagrangian with respect to Ω, the hypercomplex connection ∇ defined by Equation* (3.8) *induces a torsion-free flat L-connection $\nabla|_L$ on L.*

Proof. For all $\xi,\eta,\zeta\in\Gamma(L)$, as L is involutive and isotropic with respect to Ω, we have $\Omega(\xi,\zeta)=\Omega(\xi,\eta\circ\zeta)=\Omega(\eta\circ\xi,\zeta)=0$. By Equation (5.1), we have

$$\Omega(\nabla_\xi\eta,\zeta)=(\mathcal{L}_\eta\Omega)(\xi,\zeta)=\rho(\eta)\Omega(\xi,\zeta)-\Omega(\xi,\eta\circ\zeta)-\Omega(\eta\circ\xi,\zeta)=0.$$

As L is a Lagrangian subalgebroid with respect to Ω, we get $\nabla_\xi\eta\in\Gamma(L)$ for all $\xi,\eta\in\Gamma(L)$.

On the other hand, for all $\xi,\eta\in\Gamma(L)$, we have $\langle\xi,\eta\rangle=0$, $\langle I\xi,\eta\rangle=\langle\Omega^\sharp\xi,\eta\rangle=\Omega(\xi,\eta)=0$, $\langle J\xi,\eta\rangle=-i\langle\xi,\eta\rangle=0$, and $\langle K\xi,\eta\rangle=-i\langle\Omega^\sharp\xi,\eta\rangle=-i\Omega(\xi,\eta)=0$. It follows from Equations (3.1), (3.2), (3.3), and (3.10) that $\nabla|_L$ is a torsion-free L-connection on L and from Lemma 5.1 that $\nabla|_L$ is flat. ☐

The following lemma, which demonstrates the relation between Dirac structures and Lagrangian Lie subalgebroids, is easy to verify.

Lemma 5.6. *Let (I,J,K) be a hypercomplex structure on a Courant algebroid $(E,\rho,\langle,\rangle,\circ)$, and let $\Omega\in\Gamma(\wedge^2 L_J)$ be the associated holomorphic symplectic form relative to J. If D is a Dirac subbundle of E stable under I, J, and K, then $L=\frac{1+iJ}{2}D$ is a subalgebroid of L_J^* Lagrangian with respect to Ω.*

As consequences of Theorem 5.5 and Lemma 5.6, we consider the following two special cases.

Let M be a complex manifold (with complex structure j), and let ω be a holomorphic symplectic 2-form on M. Set $J = \left(\begin{smallmatrix} -j & 0 \\ 0 & j^* \end{smallmatrix}\right)$ (contrast Example 4.16) and $\Omega = \omega + \pi$ with $\pi = -(\overline{\omega})^{-1}$. As shown in Example 3.19, if S is a complex Lagrangian foliation, then $D = T_S \oplus T_S^{\perp}$ is a Dirac structure stable under I, J, and K. Hence, the Lie subalgebroid $L = \frac{1+\sqrt{-1}J}{2} D = T_S^{1,0} \oplus (T_S^{\perp})^{0,1}$ of $L_J^* = T^{1,0} \oplus (T^{0,1})^*$ is Lagrangian with respect to $\Omega = \omega + \pi$ according to Lemma 5.6. By Theorem 5.5, the restriction of the hypercomplex connection ∇ (defined by Equation (3.8)) to L is a flat torsion-free connection that satisfies

$$\nabla_X Y = \omega^{-1}(\imath_X \partial(\omega(Y))) \in \Gamma(T_S^{1,0}), \quad \forall X, Y \in \Gamma(T_S^{1,0}).$$

Thus, we recover the flat torsion-free connection obtained by Behrend & Fantechi in [2].

Corollary 5.7. *Let S be a complex Lagrangian foliation of a holomorphic symplectic manifold $(M; j, \omega)$. Then*

$$\nabla_X Y = \omega^{-1}(\imath_X \partial(\omega(Y))) \in \Gamma(T_S^{1,0}), \quad \forall X, Y \in \Gamma(T_S^{1,0})$$

defines a torsion-free flat $T_S^{1,0}$-connection on $T_S^{1,0}$.

Now we consider another special case. As in Example 4.15, consider a complex manifold M (with complex structure j), a holomorphic symplectic structure $\Omega = \theta \in \Omega^{0,1}(T^{1,0})$ on $T \oplus T^*$ with respect to $J = \left(\begin{smallmatrix} j & 0 \\ 0 & -j^* \end{smallmatrix}\right)$, and the associated hypercomplex triple (i, j, k) on X. As shown in Example 3.18, if S is a hypercomplex foliation, then $D = T_S \oplus T_S^{\perp}$ is a Dirac structure stable under I, J, and K. Hence, the Lie subalgebroid $L = \frac{1+\sqrt{-1}J}{2} D = T_S^{0,1} \oplus (T_S^{\perp})^{1,0}$ of $L_J^* = T^{0,1} \oplus (T^{1,0})^*$ is Lagrangian with respect to $\Omega = \theta$ according to Lemma 5.6. By Theorem 5.5, the restriction of the hypercomplex connection ∇ (defined by Equation (3.8)) to L is a flat torsion-free connection that satisfies

$$\nabla_{\overline{X}} \overline{Y} = -\bar{\theta}(\imath_{\overline{X}} \bar{\partial}(\theta(\overline{Y}))) \in \Gamma(T_S^{0,1}), \quad \forall \overline{X}, \overline{Y} \in \Gamma(T_S^{0,1}). \tag{5.5}$$

If we consider the conjugation of Equation (5.5), then we get the following corollary.

Corollary 5.8. *Let S be a hypercomplex foliation on a hypercomplex manifold $(M; i, j, k)$, and let $\theta \in \Omega^{0,1}(T^{1,0})$ (relatively to j) be defined by $\theta^{\sharp} = \frac{1}{2}(i + \sqrt{-1}k)$. Then*

$$\nabla_X Y = -\theta(\imath_X \partial(\bar{\theta}(Y))) \in \Gamma(T_S^{1,0}), \quad \forall X, Y \in \Gamma(T_S^{1,0})$$

defines a torsion-free flat $T_S^{1,0}$-connection on $T_S^{1,0}$.

References

1. James Barton and Mathieu Stiénon, *Generalized complex submanifolds*, Pacific J. Math. **236** (2008), no. 1, 23–44. MR 2398985 (2010b:53148)
2. Kai Behrend and Barbara Fantechi, *Gerstenhaber and Batalin-Vilkovisky structures on Lagrangian intersections*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston Inc., Boston, MA, 2009, pp. 1–47. MR 2641169 (2011c:53194)
3. Zhuo Chen, Mathieu Stiénon, and Ping Xu, *Geometry of Maurer-Cartan elements on complex manifolds*, Comm. Math. Phys. **297** (2010), no. 1, 169–187. MR 2645750
4. Theodore James Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 631–661. MR 998124 (90m:58065)
5. Marco Gualtieri, *Generalized complex geometry*, 2004, arXiv:math/0401221.
6. ———, *Generalized complex geometry*, 2007, arXiv:math/0703298.
7. Nigel Hitchin, *Monopoles, minimal surfaces and algebraic curves*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 105, Presses de l'Université de Montréal, Montreal, QC, 1987. MR 935967 (89h:58043)
8. ———, *Generalized geometry—an introduction*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., vol. 16, Eur. Math. Soc., Zürich, 2010, pp. 185–208. MR 2681591 (2011j:53172)

9. Yvette Kosmann-Schwarzbach, *Nijenhuis structures on Courant algebroids*, Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 4, 625–649. MR 2861782
10. Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu, *Holomorphic Poisson manifolds and holomorphic Lie algebroids*, Int. Math. Res. Not. IMRN (2008), Art. ID rnn 088, 46. MR 2439547 (2009i:53082)
11. Zhang-Ju Liu, Alan Weinstein, and Ping Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), no. 3, 547–574. MR 1472888 (98f:58203)
12. Kirill C. H. Mackenzie and Ping Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. **73** (1994), no. 2, 415–452. MR 1262213 (95b:58171)
13. Morio Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap. J. Math. **26** (1956), 43–77. MR 0095290 (20 #1796a)
14. Dmitry Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, ProQuest LLC, Ann Arbor, MI, 1999, Thesis (Ph.D.)–University of California, Berkeley. MR 2699145
15. Pavol Ševera and Alan Weinstein, *Poisson geometry with a 3-form background*, Progr. Theoret. Phys. Suppl. (2001), no. 144, 145–154, Noncommutative geometry and string theory (Yokohama, 2001). MR 2023853 (2005e:53132)
16. Mathieu Stiénon, *Hypercomplex structures on Courant algebroids*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 9-10, 545–550. MR 2576906
17. Ping Xu, *Hyper-Lie Poisson structures*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 3, 279–302. MR 1443488 (98f:53044)
18. Kentaro Yano and Mitsue Ako, *An affine connection in an almost quaternion manifold*, J. Differential Geometry **8** (1973), 341–347. MR 0355892 (50 #8366)

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